A SURVEY OF QUASI-BRITTLE FRACTURE CRACK THEORY

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My purpose in the present survey is to trace the development of the basic concepts of quasi-brittle fracture crack theory and to outline the present state of the art. Selecting the most relevant material from the massive literature dealing with various theoretical and experimental aspects of fracture theory and in some way concerned with the propagation of macroscopic cracks proved a very difficult task. This prevented me from giving adequate coverage to experimental studies, dynamics problems, the beam approximation, and several other aspects of theory and applications. I am well aware of the fact that the survey contains gaps and that some studies have either been overlooked completely or not discussed with adequate thoroughness. This applies in large measure to the numerous reports and theses from various foreign universities and laboratories which were not available to me. Where possible, I have tried to convey the original interpretations placed by authors on their own results. Notation has been altered as required for the sake of consistency.

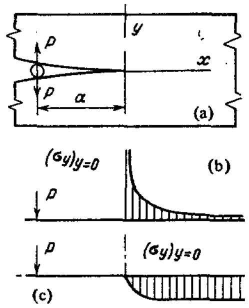

Fig. 1

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I. DEVELOPMENT OF THE BASIC CONCEPTS

§1. Mathematical foundations of the theory. The theory of quasi-brittle fracture is based on the elasticity theory for small strains. The most interesting results relevant to our subject are certain solutions of elasticity theory for a plane weakened by a straight-line crack or cracks; the analysis of the stress and strain states near a crack tip was also an essential part of the preliminary work.

In 1909 G. V. Kolosov [1] obtained the solution for a plane weakened by an elliptical hole under uniform tension. The solution of the problem of a plane with a straight-line crack is a special case of Kolosov's solution. In 1913 the same solution was obtained by Inglis [1].

N. I. Muskhelishvili [1] (1919) solved the problem of a plane with an elliptical hole in an arbitrary stress field.

In subsequent studies Kolosov, Muskhelishvili, and their followers developed the mathematical apparatus of elasticity theory which permitted solution of the basic two-dimensional problems of elasticity theory. The resulting solutions played a major role in the development of the theory of quasi-brittle fracture.

In 1939 Westergaard [2] published his paper on the theory of contact problems and cracks in elastic bodies, which became widely known in the West. Westergaard introduced the function of a complex variable

$$Z = Z(z) = Z(x + iy) = \operatorname{Re} Z + i \operatorname{Im} Z. \qquad (1.1)$$

He defined the functions Z', $\bar{Z}$, $\bar{\bar{Z}}$, as the derivative and the first and second integrals of Z,

$$Z' = \frac{dZ}{dz}, \quad Z = \frac{d\bar{Z}}{\partial z}, \quad \bar{Z} = \frac{d\bar{\bar{Z}}}{dz}. \qquad (1.2)$$

The stresses in certain important groups of problems can be written in the form*

$$\sigma_x = \operatorname{Re} Z - y \operatorname{Im} Z', \quad \sigma_y = \operatorname{Re} Z + y \operatorname{Im} Z',$$
$$\tau_{xy} = -y \operatorname{Re} Z'. \qquad (1.3)$$

Relations (1.3) satisfy the equilibrium equations of the planar problem. The solutions obtained from Eqs. (1.3) satisfy the condition

$$\sigma_x = \sigma_y, \quad \tau_{xy} = 0 \quad \text{for} \quad y = 0. \qquad (1.4)$$

The first condition of (1.4) is valid over continuous domains.

The displacement u, v along the x- and y-axis are given by

$$2Gu = (1 - 2\nu) \operatorname{Re} \bar{Z} - y \operatorname{Im} Z,$$
$$2Gv = 2(1 - \nu) \operatorname{Im} \bar{Z} - y \operatorname{Re} Z \qquad (1.5)$$

where G is the shear modulus.

The displacement v along the y-axis for y = 0 can be expressed in the form

$$v_0 = \frac{1 - \nu}{G} \operatorname{Im} \bar{Z}. \qquad (1.6)$$

Taking any function of a complex variable $Z(z)$, we can use Eqs. (1.3) and (1.5) to obtain some solution for the problem of the plane theory of elasticity. This is the basic idea of the semi-inverse method of Westergaard.

Choosing appropriate functions Z, Westergaard investigated several contact and crack problems.

He proposed functions Z for the case of one internal crack and for several collinear cracks in a plane under uniform tension at infinity

$$Z = p / \sqrt{1 - a^2/z^2}, \qquad Z = p \bigg/ \sqrt{1 - \frac{\sin^2(\pi a/l)}{\sin^2(\pi z/l)}}. \qquad (1.7)$$

Westergaard noted that in order to consider the uniform pressure p applied along the banks of a crack as the load it is necessary to take functions (1.7) in the form $Z_1 = Z - p$. He also considered problems concerning a crack opened by a wedge applying the force P as shown

*The Westergaard formulas are a special case of the Kolosov-Muskhelishvili formulas for $\Phi(z) = Z/2$, $\Psi(z) = zZ/2$. Several of the problems investigated by Westergaard had already been considered by Muskhelishvili in the second edition of his monograph Some Basic Problems of the Mathematical Theory of Elasticity (1935). Westergaard's results are presented here because they were used in this form by Irwin and other developers of the theory of quasi-brittle fracture.

in Fig. 1. The functions considered by Westergaard in this case were

$$Z = \frac{P}{\pi (a+z)} \sqrt{\frac{a}{z}}, \qquad Z = -\frac{P}{\pi (a+z)} \sqrt{\frac{z}{a}}. \quad (1.8)$$

Figures 1b and 1c show the corresponding stress curves for y = 0.

Westergaard suggested that the shape of the second curve (Fig. 1c) with finite stresses at the tip of the crack be regarded as typical of brittle materials such as concrete. Finally, the function

$$Z = Pa/\pi z \sqrt{z^2 - a^2} \quad (1.9)$$

defines the stress-strain state of a plane with a central crack opened by concentrated forces P (Fig. 2).

Sneddon [1, 2], Sneddon and Elliot [1], et al. further extended the mathematical methods of the theory of cracks in elastic media.

Sneddon [1] (1946) considered the plane and axisymmetric problems of isolated cracks in infinite bodies. For the planar problem he used Westergaard's solution [2] and analyzed the stress state near a crack of length $2a$ acted on by a constant internal pressure p_0. He chose Z as

$$Z = p_0 \left(\frac{z}{\sqrt{z^2 - a^2}} - 1 \right). \quad (1.10)$$

Among other things, Sneddon determined the stress distribution near the edge of the crack.

Let us review Sneddon's analysis briefly. We begin by setting

$$z = re^{i\psi}, \qquad z - a = r_1 e^{i\psi_1}, \qquad z + a = r_2 e^{i\psi_2} \quad (1.11)$$

as indicated in Fig. 3.

From Eqs. (1.3), (1.10), and (1.11) we obtain

$$^1/_2 (\sigma_x + \sigma_y) = p_0 \left[r (r_1 r_2)^{-1/2} \cos (\psi - {}^1/_2 \psi_1 - {}^1/_2 \psi_2) - 1 \right],$$

$$^1/_2 (\sigma_y - \sigma_x) = p_0 a^2 r (r_1 r_2)^{-3/2} \sin \psi \sin {}^3/_2 (\psi_1 + \psi_2)$$

$$\tau_{xy} = p_0 a^2 r (r_1 r_2)^{-3/2} \sin \psi \cos {}^3/_2 (\psi_1 + \psi_2) \quad (1.12)$$

To determine the stress distribution near the right-hand tip of the crack, we set $r_1 = \delta$, where δ is a small parameter. Next, we set $\psi_1 = \theta$. To within higher-order terms we have

$$r = a + \delta \cos \theta, \quad r_2 = 2a + \delta \cos \theta,$$

$$\theta = \delta \sin (\theta/a), \quad \theta_2 = {}^1/_2 \delta \sin (\theta / 2a) \quad (1.13)$$

Linearizing (1.12), (1.13) and carrying out some transformations, we obtain

$$\sigma_x = p_0 (a/2\delta)^{1/2} ({}^3/_4 \cos \theta/2 + {}^1/_4 \cos 5\theta/2),$$

$$\sigma_y = p_0 (a/2\delta)^{1/2} ({}^5/_4 \cos \theta/2 - {}^1/_4 \cos 5\theta/2)$$

$$\tau_{xy} = p_0 (a/8\delta)^{1/2} \sin \theta \cos 3\theta/2. \quad (1.14)$$

[Barenblatt [1] (1956) investigated asymptotic expressions for the stresses and displacements near the crack tips for the problem shown in Fig. 2 with compressive forces $-q_\infty$ acting at infinity. He derived expressions for the stresses along the x-axis ($\theta = 0$ in formulas (1.14)) for the displacement v along the y-axis for small θ.]

In the same paper Sneddon used Fourier transformation to investigate the case of a circular crack under variable axisymmetric pressure. It turned out that near the edge of a circular crack acted on by a constant pressure p_0, we have (in analogous symbols)

$$\sigma_r = (2/\pi) \sigma_x, \quad \sigma_z = (2/\pi) \sigma_y, \quad \tau_{rz} = (2/\pi) \tau_{xy}$$

$$\sigma_\Phi = \frac{4\nu p_0}{\pi} \left(\frac{c}{2\delta} \right)^{1/2} \cos \frac{\theta}{2} \quad (1.15)$$

where σ_r, σ_z, τ_{rz}, and σ_Φ are the stress components in the cylindrical coordinate syster r, Φ, z. The components σ_x, σ_y and τ_{xy} are given by formulas (1.14).

We must also take note of Sneddon's analysis of the elastic energy of a cracked body.

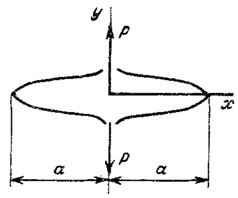

Fig. 2

In 1946 Sneddon and Elliot [1] discussed the stress distribution near a plane Griffith crack under a pressure, variable along the length of the crack, applied to its banks. They availed themselves of the Fourier cosine transform and the results of dual integral equation theory. Their results were entirely analogous to those of Sneddon [1] for the axisymmetric problem.

Sneddon later cited similar results for the plane and the axisymmetric problems of elasticity theory in his monographs [2, 3].

Williams [1, 2] (1952) considered the general solution of elasticity theory near the vertex of an infinite sector of the angle α (Fig. 4a). He later investigated the special case of his solution when $\alpha = 2\pi$ and both banks of the crack are stress-free (Fig. 4b) [3] (1957).

Using the polar coordinate system r, θ, Williams shows (see the corrected variant of his formulas in [5]) that the biharmonic stress function $\chi(r, \theta)$ can be resolved into its even and odd parts with respect to the y-axis, $\chi_e(r,\theta)$ and $\chi_O(r,\theta)$, respectively, i.e.,

$$\chi_e(r, \theta) = \sum_{n=1,2,3} \Big\{ (-1)^{n-1} a_{2n-1} r^{n+1/2} \Big[-\cos (n - {}^3/_2) \theta + \frac{2n-3}{2n+1} \cos \left(n + \frac{1}{2} \right) \theta \Big] + (-1)^n a_{2n} r^{n+1} \left[-\cos (n-1) \theta + \cos (n+1) \theta \right] \Big\}, \quad (1.16)$$

$$\chi_0(r, \psi) = \sum_{n=1,2,3\ldots} \Big\{ (-1)^{n-1} b_{2n-1} r^{n+1/2} + [\sin (n - {}^3/_2) \theta - \sin (n + {}^1/_2) \theta] + (-1)^n b_{2n} r^{n+1} \Big[-\sin (n-1) \theta + \frac{n-1}{n+1} \sin (n+1) \theta \Big\}. \quad (1.17)$$

Functions (1.16) and (1.17) define the stress field satisfying the equilibrium and boundary conditions at the free banks of the crack. The constants a_i and b_i must be determined from the boundary conditions at infinity in the case of an infinite sector, and from those at the fixed boundaries in the case of a body of finite size.

Expressions (1.16) and (1.17) imply that the stresses near the edge of the crack can be written as

$$\sigma_r = \frac{1}{4r^{1/2}} \Big\{ a_1 \Big[-5 \cos \frac{\theta}{2} + \cos \frac{3\theta}{2} \Big] + b_1 \left[-5 \sin \theta/2 + 3 \sin 3\theta/2 \right] \Big\} + 4a_2 \cos^2 \theta + 0 (r^{1/2}), \quad (1.18)$$

$$\sigma_\theta = \frac{1}{4r^{1/2}} \Big\{ a_1 \Big[-3 \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} \Big] + b_1 \Big[-3 \sin \frac{\theta}{2} - 3 \sin \frac{3\theta}{2} \Big] \Big\} + 4a_2 \sin^2 \theta + 0 (r^{1/2}), \quad (1.19)$$

$$\tau_{r\theta} = \frac{1}{4r^{1/2}} \Big\{ a_1 \Big[-\sin \frac{\theta}{2} - \sin \frac{3\theta}{2} \Big] + b_1 \Big[\cos \frac{\theta}{2} + 3 \cos \frac{3\theta}{2} \Big] \Big\} - 2a_2 \sin 2\theta + 0 (r^{1/2}). \quad (1.20)$$

It is clear that in Cartesian coordinates the terms containing the coefficient italic a_2 in (1.18)–(1.20) correspond to the purely tensile state

$$\sigma_x = 4a_2, \qquad \sigma_y = 0, \qquad \tau_{xy} = 0. \tag{1.21}$$

Since most cases do not involve pure tension along the crack edge, Williams assumes at this point that $a_2 = 0$.

Thus, the variation of the stress components over the radius is generally of the order of $r^{-1/2} + 0(r^{1/2})$.

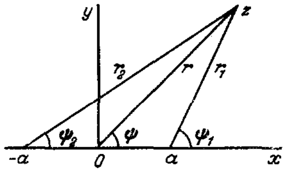

Fig. 3

Williams now proceeds to determine the total potential energy density W, obtaining

$$\begin{aligned} 32ErW = a_1^2\Big[(34-30\nu)\cos^2\frac{\theta}{2} + \\ + 2(1+\nu)\sin^2\frac{\theta}{2} + 2(1+\nu) - \\ - 4(1+\nu)\cos 2\theta\Big] + \Big[b_1^2(30-34\nu)\sin^2\frac{\theta}{2} + \\ + 2(1+\nu)\cos^2\frac{\theta}{2} + 18(1+\nu) + 12(1+\nu)\cos 2\theta\Big] + \\ + 2a_1b_1\Big[(32-22\nu)\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \\ - 8(1+\nu)\sin 2\theta\Big] + \cdots \end{aligned} \tag{1.22}$$

His expressions for the strain energy of distortion density W_d is

$$\begin{aligned} 2GrW_d = a_1^2\left(1 + 3\sin^2\frac{\theta}{2}\right)\cos^2\frac{\theta}{2} + \\ + 2b_1^2\left[3 + \left(1 + 3\cos\frac{\theta}{2}\right)\left(1 - 3\cos\frac{\theta}{2}\right)\sin^2\frac{\theta}{2}\right] + \\ + 2a_1b_1\sin\theta + \ldots \end{aligned} \tag{1.23}$$

He then obtains the following expressions for the displacement components:

$$\begin{aligned} 2Gu_r = r^{1/2}\Big\{a_1\Big[\Big(-\frac{5}{2} + 4\gamma\Big)\cos\frac{\theta}{2} + \frac{1}{2}\cos\frac{3\theta}{2}\Big] + \\ + b_1\Big[\Big(-\frac{5}{2} + 4\gamma\Big)\sin\frac{\theta}{2} + \frac{3}{2}\sin\frac{3\theta}{2}\Big]\Big\} + \cdots \end{aligned} \tag{1.24}$$

$$\begin{aligned} 2Gu_\theta = r^{1/2}\Big\{a_1\Big[\Big(\frac{7}{2} - 4\gamma\Big)\sin\frac{\theta}{2} - \frac{1}{2}\sin\frac{3\theta}{2}\Big] + \\ + b_1\Big[-\Big(\frac{7}{2} - 4\gamma\Big)\cos\frac{\theta}{2} + \frac{3}{2}\cos\frac{3\theta}{2}\Big]\Big\} + \ldots \end{aligned} \tag{1.25}$$

where u_r and u_θ are the displacement components in the polar coordinate system, and $\gamma = \nu/(1+\nu)$.

Next, Williams carries out separate investigations of the symmetric ($b_i = 0$) and antisymmetric ($a_i = 0$) stress distributions with respect to the y-axis.

In the symmetric case the character of the singularity near the tip of the crack is the same as that of the singularity defined by Inglis, Westergaard, et al. Williams notes that since $\tau_{r\theta} = 0$ for $\theta = 0$ and $\sigma_r \to \sigma_\theta = -a_1 r^{-1/2}$, the tip of a planar crack tends strongly to a state of uniform two-dimensional (hydrostatic) pressure which makes it possible to use elasticity analysis near the crack tip despite the proportionality of the stress components to $r^{-1/2}$. He then determines the difference between the principal stresses near the crack tip and shows that the tendency towards hydrostatic tension diminishes with the distance from the crack tip.

Williams also determined the variations of the principal stresses along a circle with a small radius $r = r_0$ and whose center lies at the crack tip (Fig. 5a). The maximum tensile stress is attained at $\theta = \pm\pi/3$. The maximum shearing stress corresponds to $\theta = \pi/2$ and is equal to one-half of the hydrostatic tension acting at $\theta = 0$ in the same radial direction. Figure 6 shows the magnitudes and orientations of the principal stresses for the characteristic directions, $\sigma_0 = -a_1 r_0^{1/2}$. The maximum strain energy of distortion lies not on the line of the crack, but rather in the direction $\theta = \pm\cos^{-1}(1/3) \approx \pm 70°$ where it is larger by one-third (see Fig. 7a).

Williams notes that for $a_1 = 0$ we arrive at the problems considered by Westergaard [1, 2] in which the stress concentration at the crack tip is negligible.

Williams carried out a similar analysis in the antisymmetric case. The stress singularity was once again of the order of $r^{-1/2}$. The variation of the principal stresses is shown in Fig. 5b; the variation of the strain energy of distortion density appears in Fig. 7b.

The results of the above studies imply that the asymptotic behavior of the stresses and strains near the crack tip does not depend on the form of the stress state. The characteristic dimensions of the crack and of the body, the loads, and the other parameters constituting the initial conditions determine only the coefficients a_1 and b_1 in (1.18)–(1.20) characterizing the stress intensity and the strains near the crack tip.

§2. The principles of the Griffith-Irwin theory. The originator of the mathematical theory of crack propagation in elastic media and of quasi-brittle fracture theory was Griffith [1, 2]. In studying the fracture of glass Griffith assumed that it contained crack-like defects. According to him, the largest of these cracks becomes self-propagating when the rate of propagation of the strain energy exceeds the rate of increase of the surface energy of the crack during extension.

Griffith's argument can be stated in the following form. Let us consider an ideally elastic body (a discontinuity surface or a cut of zero thickness) of area S. We assume that the body is strained by some system of external body forces F_i and surface forces p_i. We assume, moreover, that the outer boundary of the body is fixed and that the displacement discontinuity surface increases by δS. The increment δS corresponds to the rupture of internal bonds in the elastic body. The work done by the external forces in the case of fixed boundaries is equal to zero, and the elastic energy of the body decreases by the amount δW. The quantity $\delta W/\delta S$ has come to be called the rate of liberation of elastic energy during crack propagation.

At the same time, the increases in crack surface increase the crack surface energy by $\delta\pi$; similarly, $\delta\pi/\delta S$ is the rate of increase of the surface energy of the body during propagation of the crack. According to Griffith, the energy criterion for the equilibrium state of the crack can be written as

$$\frac{\delta(W - \Pi)}{\delta S} = 0. \tag{1.26}$$

The crack is stable when

$$\frac{\delta^2(W - \Pi)}{\delta S^2} > 0. \tag{1.27}$$

and unstable when

$$\frac{\delta^2(W - \Pi)}{\delta S^2} < 0. \tag{1.28}$$

Griffith assumed that

$$\delta\Pi = 2T\delta S, \qquad T = \text{const} \tag{1.29}$$

where T is the surface tension.

We can therefore write condition (1.26) in the form

$$\frac{\delta W}{\delta S} = 2T. \qquad (1.30)$$

For example, Griffith considered a surface with a straight-line cut of length $2l$ under the uniform omnilateral tension p at infinity. Using the Inglis solution [1], Griffith obtained the critical values of p_0 for the planar strain and planar stress states

$$p_0 = \sqrt{\frac{2ET}{\pi(1-\nu^2)l}}, \quad p_0 = \sqrt{\frac{2ET}{\pi l}} \qquad (1.31)$$

where E is the elastic modulus and ν is the Poisson coefficient.

The solution of the problem of elasticity theory concerning a straight-line crack in a plane under tension has the following peculiarity: for any arbitrarily small but finite load p_0 the contour of the straight-line crack is deformed into an elliptical cavity, and the stresses at the crack tips turn out to be arbitrarily large. Such singularities are generally involved in the solutions of the equations of linear elasticity theory in cases when the geometric or force boundary conditions have singularities. This is exemplified by the solutions of linear elasticity theory concerning the impression of dies with corners, the action of concentrated forces, the presence of notches in the boundary of a body, etc. In the case of the Kolosov-Inglis problem such a singularity occurs at the tips of the crack, where the radius of curvature is zero and the curvature is infinite.

We know that such singularities of the solution have no analogs in the physical behavior of real media, but that they correctly describe the character of the stress and strain states of a body in a sufficiently small neighborhood.

Griffith also considered a crack in the form of an elliptical cavity with a small finite radius of curvature. However, his attempt to avoid infinitely large stresses near the crack tips and to relate the notions of continuum mechanics to those of solid state physics were unsuccessful. According to his estimates, the radius of curvature at the crack tip would have to be a quantity on the order of interatomic distances, whereas the continuum mechanics deals with distances of much higher order of magnitude.

However, it turned out to be unnecessary to relate the notions of continuum mechanics to the molecular and atomic notions of solid-state physics in this case.

Further investigation showed that modifications of Griffith's concepts could be used to describe a sufficiently broad class of solid-body fractures within the framework of phenomenological theory.

Griffith checked his results experimentally.

After the appearance of Griffith's first paper, Smekal [1] published an analysis of the problem of quasi-brittle fracture. Specifically, he corrected certain inaccuracies present in Griffith's solution (for references to other papers by Smekal see Weiss and Yukawa [1]).

Wolf [1] reviewed Griffith's ideas and solved the problem of crack propagation for a straight-line slit in three cases: omnilateral uniform tension at infinity (see Fig. 8a; this was the case considered by Griffith), uniaxial uniform tension at infinity (Fig. 8b), and pure bending (Fig. 8c).

In an addendum to his paper Wolf discussed the problem of fracture of a body containing many differently oriented small cracks of length $2l$. Assuming that the stress state was the combination of the three stress states which he had considered, he formulated the following condition of cracking of the body:

$$4\nu\sigma_2 + (1 - 3\nu)(\sigma_2 - \sigma_1)\,\sigma_2 = 4TE/\pi l \qquad (1.32)$$

where σ_1, σ_2 are the principal stresses at sufficient distances from the crack.

Wolf suggested that the quantity $4TE/\pi l$ be regarded as a material constant. His results are discussed in a recent paper by Swedlow [1]. An attempt to construct a theory of strength on the basis of material fissility was also made by Mossakovskii and Rybka [1].

Certain important results which had a considerable effect on the theory of quasi-brittle fracture were obtained by Neuber [1].

Neuber did not concern himself with cracking theory as such, but rather with stress concentrations near various recesses for several planar and axisymmetric problems, as well as for torsion problems of elasticity theory.

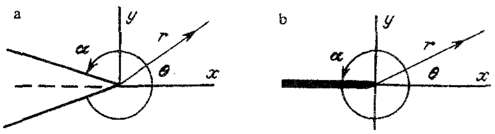

Fig. 4

Making use of the semi-inverse methods of elasticity theory, Neuber based his investigations on the general solution in the appropriate coordinate system.* Despite the well-known cumbersomeness of his method, Neuber succeeded in obtaining a series of effective solutions of important problems.

Solutions for bodies weakened by various types of slits and notches follow as limiting cases from the Neuber solutions.

For deep smooth grooves (Fig. 9a) the maximum stress occurs at the vertex of the groove, and the concentration coefficient α can be written as

$$\alpha = f\,(a/\rho) \qquad (1.33)$$

where a is one-half of the width of the narrowest transverse cross section and ρ is the minimum radius of curvature of the groove.

In the theory of grooves with sharp corners, Neuber encountered the singularities of the solutions of elasticity theory which occur in the mathematical theory of cracks. With a sufficiently small radius of curvature the formally computed stress concentration coefficient assumes arbitrarily large values devoid of practical meaning. This led Neuber to suggest the introduction of a certain particle having other than elastic properties in a sufficiently small neighborhood of the groove. The size of this particle assumes the role of a new material constant. Once this constant has been determined experimentally, the practically possible error of determining the concentration coefficient in the remainder of the zone is quite acceptable.

Let us consider Fig. 9b. Near the end of the notch we have a "Neuber particle" of width ε. Let us assume that the solution of the classical theory of elasticity has been found. We assume that the particle is so small that the stresses acting on its surface can be considered constant. The new values of the stresses in the particle are obtained by averaging the old values at its surface. The concentration coefficient then becomes

$$\alpha = f\,(2a/\varepsilon). \qquad (1.34)$$

Thus, according to (1.33) and (1.34), the role of the radius of curvature in the case of a sharp-cornered groove is played by the quantity $\varepsilon/2$. Setting $\rho' = \varepsilon/2$, Neuber noted that the constant ρ' is related to the structure of the material, assuming different values for different materials. For example, the ρ' for steel is approximately 0.5 mm.

According to Neuber there is another way of explaining the decrease in the stress concentration coefficient: instead of considering the stress state relative to the unstrained state in accordance with linear elasticity theory, one can use the theory of finite strains to refer the equilibrium equations to the strained state.** But since allowance for the effect of straining and introduction of an appropriate "particle" to allow for the structure of the material has the effect of decreasing the stress concentration in a highly curved groove, we can use one of the above approaches, e.g. that of Neuber, provided we choose our constant ρ' in accordance with experimental data. Thus, the introduction of ρ' is in a sense equivalent to allow finite strains near the end of a sharp-cornered groove.

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*An analogous solution was obtained before Neuber by Grodskii and Papkovich in their Theory of Elasticity, Oborongiz, pp. 120–131, 1939.

**This was, in fact, the approach used by Föppl (Ing. Arch., Vol. 7, p. 229, 1936).

Neuber showed that the linear theory of elasticity yields acceptable results in the immediate neighborhood of the vertex of a sharp-cornered groove.

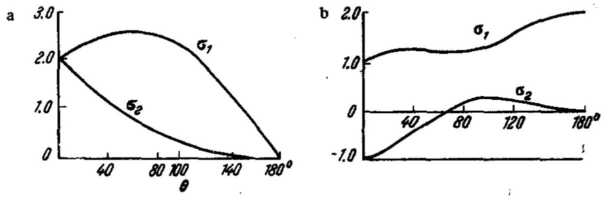

Fig. 5

Irwin [1] (1948) and Orowan [1] (1950) both expressed the view that the Griffith theory can be used to describe the cracking of a whole range of materials.* It turns out that certain metals which exhibit considerable plastic properties experience brittle fracture if they contain cracks.

This is due to the fact that the stress state near the tips of cracks or notches is close to uniform triaxial tension. Brittle fracture in ordinary malleable metals is always accompanied by small plastic strains in a thin layer at the fracture surface. The elastic energy of the body during the fracture of such materials is converted not only into surface energy, but also into the work expended on the formation of plastic strains in the surface layer of the crack. According to Orowan's estimates, for example, the latter quantity in the case of iron is 10^3 times larger than the surface tension energy, which means that the latter can be neglected. In the case of high-carbon steel the two quantities are comparable.

Let us rewrite relation (1.30) as

$$\frac{\delta W}{\delta S} = \mathcal{G}. \qquad (1.35)$$

According to the Castigliano theorem of elasticity theory, the force is equal to the partial derivative of the energy with respect to the corresponding displacement, so that relation (1.35) can be considered from this standpoint. In this case the quantity $\mathcal{G}$ can be interpreted as the force per unit area (or length) of the crack edge which acts to propagate the crack.

According to Irwin and Orovan, there exists a certain limiting value $\mathcal{G}_c$, which is a constant of the material. Its attainment coincides with the onset of crack propagation.

The crack propagation criterion therefore becomes

$$\mathcal{G} \geqslant \mathcal{G}_c. \qquad (1.36)$$

Cracks are stable if

$$\mathcal{G} < \mathcal{G}_c. \qquad (1.37)$$

Experimental determination of $\mathcal{G}_c$ is discussed in numerous papers, e.g., Irwin [1], Orlov [1, 2], Irwin and Kies [1, 2], Irwin, Kies, and Smith [1], et al. (A more detailed bibliography will be found in the collection Fracture Toughness Testing and Its Applications, Philadelphia, 1965.)

Irwin's paper [3] was a landmark in the development of the mathematical theory of cracks. Its findings are developed in his subsequent publications.

In [3] Irwin used the semi-inverse method of Westergaard together with functions (1.7) to obtain three new solutions. The first of these applies to a crack of length $2a$ lying along the x-axis and acted on by the cleaving (opening) forces P applied at the points x = b (Fig. 10). The corresponding Westergaard function is of the form

$$Z = \frac{Pa}{\pi (z-a) z}\left[\frac{1-(b/a)^2}{1-(b/z)^2}\right]^{1/2}. \qquad (1.38)$$

*Irwin [1] noted that the relationship between Griffith's ideas and the quasi-brittle fracture of metals was first considered by Zener and Hollomon, Trans. Am. Soc. Metals, Vol. 33, pp. 163–235, 1944.

The second example preserves all the conditions of the latter case and differs by the addition of the cleaving forces P at the points x = = –b; the Westergaard function is

$$Z = \frac{Pa}{\pi (z^2-a^2)}\left[\frac{1-(b/a)^2}{1-(b/z)^2}\right]^{1/2}. \qquad (1.39)$$

The third example concerns a periodic system of cracks of length $2a$ whose centers are a distance l apart, and where the cleaving forces P are applied at the crack centers. The Westergaard function is of the form

$$Z = \frac{P \sin (\pi a/l)}{l \sin^2 (\pi z/l)}\left[1 - \frac{\sin^2 (\pi a/l)}{\sin^2 (\pi z/l)}\right]^{-1/2}. \qquad (1.40)$$

In the same paper Irwin discussed the applicability of solution (1.40) to a plate of finite length l.

Like Sneddon [1], Irwin used the substitution of variables

$$z = a + re^{i\theta}, \quad r^2 = (x-a)^2 + y^2, \quad \mathrm{tg}\,\theta = y/(x-a) \qquad (1.41)$$

in all five examples.

Confining himself to a small neighborhood at the crack tip and assuming that the quantities r/a and $r/(a - b)$ are negligible compared with unity, Irwin (see Sneddon [1]) obtained

$$\sigma_x = \left(\frac{E\mathcal{G}}{\pi}\right)^{1/2} \frac{\cos \theta/2}{\sqrt{2r}}\left(1 - \sin\frac{\theta}{2} \sin\frac{3\theta}{2}\right), \ldots \qquad (1.42)$$

where $\mathcal{G}$ is a quantity which depends on the load (he had not yet established the relationship between the $\mathcal{G}$ in (1.42) and (1.35).

Irwin's expansions (1.42) differ from Sneddon's expansions (1.14) and from the first term of Williams' expansion (1.18)–(1.20) in notation only and describe the same stress state near the crack tip.

Irwin defined the quantity $(E\mathcal{G}/\pi)^{1/2}$ as the stress intensity coefficient. He then introduced the following symbol for the stress intensity coefficient in the case of transverse-rupture cracks:

$$K_{\mathrm{I}} = (E\mathcal{G}/\pi)^{1/2}. \qquad (1.43)$$

Next, he derived a formula which enabled him to establish the identity of the quantities $\mathcal{G}$ appearing in (1.42) and (1.35) and thus to formulate a force approach in crack theory equivalent to the energy approach of Griffith. His reasoning was as follows.

Let the origin be shifted to the right-hand tip of the crack as shown in Fig. 11. We assume that the crack has propagated to the value x = = α. We also assume that α is much smaller than the length of the crack. If the forces acting along the x-axis are defined as

$$S_y(q) = q\left(\frac{E\mathcal{G}}{\pi}\right)^{1/2} \frac{1}{\sqrt{2x}}, \quad 0 \leqslant x \leqslant \alpha \qquad (1.44)$$

where q is a parameter, and if they are applied to the crack edges, then the crack closes over the segment $0 \le x \le \alpha$ as the parameter q varies from zero to unity.

The closing segment of the crack can (in the same approximation) be written as

$$v(q) = (1-q)\frac{2}{E}\left(\frac{E\mathcal{G}}{\pi}\right)^{1/2} \sqrt{2(\alpha - x)}. \qquad (1.45)$$

Hence, the work required to close a crack over the above segment is given by

$$\int_0^\alpha S_y(1)\, v(0)\, dx = \frac{2\mathcal{G}}{\pi}\int_0^\alpha \left(\frac{\alpha - x}{x}\right)^{1/2} dx = \alpha\mathcal{G}. \qquad (1.46)$$

The amount of work defined by (1.46) is equal to the change δW in the elastic energy of the body. The role of the crack length variation δl is played by the quantity α. This implies that expression (1.46) is fully equivalent to (1.35) for the opening of cracks in the planar case.

Irwin noted that the above analysis was based on the relations of the linear theory of elasticity, while crack zones experience such effects as local stress relaxation and distortion of the opening crack through plastic flow. It should not be assumed, however, that these effects can result in a marked difference between the true rate of strain energy loss and its computed value. The method which yields relation (1.46) is equivalent to computing the derivative of the total strain energy with respect to crack length. The contribution of the tip zone to the overall energy balance is relatively small.

Citing experimental findings, Irwin wrote that if the plastic strains near a crack affect the stress field only at distances from the crack which are smaller than its length, then the effect of these plastic strains on $\mathcal{G}$ is correspondingly slight.

Irwin then discussed the problems involved in experimental determination of $\mathcal{G}_c$.

In his later paper [4] Irwin again formulated the basic principles of the theory. He computed the strain energy density Ψ in a region of radius r_1 around the edge of a circular crack, obtaining

$$\int_0^{r_1}\int_{-\pi}^{\pi} \frac{\partial \Psi}{\partial a}\, r\, dr\, d\theta = \left(\frac{5-8\nu}{8-8\nu}\right) r_1 \frac{\partial \mathcal{G}}{\partial a} \tag{1.47}$$

where a is the crack radius.

For a central crack in a sufficiently large plate under homogeneous tension the quantity $\mathcal{G}$ is proportional to a at sufficient distances away from the crack, so that

$$r_1 \frac{\partial \mathcal{G}}{\partial a} = \frac{r_1}{a}. \tag{1.48}$$

For central cracks opened by cleaving forces applied at the crack center the quantity $\mathcal{G}$ is inversely proportional to a, so that

$$r_1 \frac{\partial \mathcal{G}}{\partial a} = -\frac{r_1}{a}\mathcal{G}. \tag{1.49}$$

According to Irwin, the integral (1.47) is of the same order of magnitude as the error involved in determining $\mathcal{G}$ due to neglect of plastic strains if the zone bounded by the radius r_1 includes the major portion of the plastic straining zone. When $\mathcal{G}$ considered as a function of crack extension passes through an extremum, integral (1.47) becomes zero. By hypothesis, the plastic strains are localized in a surface layer whose thickness is small as compared with the length of the crack. The reduction of stresses by plastic flow near the crack surfaces can therefore be estimated by way of the change of $\mathcal{G}$ characterized by the derivative $\partial \mathcal{G}/\partial a$, but this reduction is relatively small. Irwin also cited experimental data for the material constant $\mathcal{G}_c$ obtained for central cracks opened in plates in planar stress states. He noted that the significance of $\mathcal{G}_c$ in quasi-brittle fracture theory is the same as that of the yield stress in plasticity theory.

In the following year there appeared a discussion of Irwin's paper [3].

McClintock noted that even if the applied stress is very large, the radius of the plastic zone is still small as compared with crack length. He obtained some appropriate estimates and then noted that the plastic zone consists of two regions: the first of these is due to the overall yield of the macroscopic domains adjacent to the notch; the second is due to the brittle fracture necessary for branching of the cracks at the nearby grains. The radius of the latter region is roughly equal to the size of a single grain.

Williams proposed an additional interpretation of the quantity $\mathcal{G}$, pointing out its relationship with the radius of curvature at the tip of the crack. The first term of the expansion of the displacement normal to the crack at its tip is of the form

$$v(r, 0) = \frac{2}{E}\left(\frac{E\mathcal{G}}{\pi}\right)^{1/2} (2x)^{1/2} \tag{1.50}$$

with the origin placed at the tip.

The local radius of curvature ρ at the base of the crack is related to the quantity $\mathcal{G}$ by the expression

$$\mathcal{G} = \frac{\pi E}{4}\rho. \tag{1.51}$$

Williams also noted the relationship between the Irwin and Neuber theories.

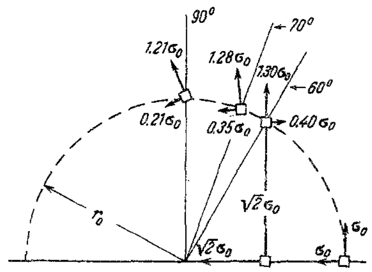

Fig. 6. The maximum tensile stress corresponds to 60°; the maximum octahedral stress to 70°, and the maximum shearing stress to 90°.

In concluding the discussion Irwin pointed out the way in which his results might be extended to the case of longitudinal shear. This can be done simply be writing the Airy stress function in the form $F = -y \operatorname{Re} \bar{Z}$, where Z is the Westergaard function. The five stress functions given by (1.7) and (1.38)-(1.40) determine the solutions of five special problems for cracks acted on by shearing forces. The crack extending force associated with each of these stress fields determines the extension of the crack as a shear dislocation.

Irwin noted that the relationship of Neuber's hypothesis of a plastic particle to the Griffith theory has been discussed by Orowan [2]. Making use of Neuber's hypothesis, Irwin obtained the following expression for the stress intensity coefficient:

$$K = \sigma_p \sqrt{\frac{\varepsilon}{2}} \tag{1.52}$$

where σ_p is the average tensile stress at the crack tip and ε is the length of the Neuber particle.

Experiments on fracture resistance do not yield σ_p or $\sqrt{\varepsilon}$ separately, so that in order to obtain the critical size of a Neuber particle one must assume that the quantity σ_p, which is usually the limiting value of the fracture resistance, is already known.

We recall that in the theory of quasi-brittle fracture it is sufficient to determine the stress intensity coefficient K.

Irwin then considered the factors involved in the experimental determination of $\mathcal{G}_c$.

In his detailed article on "Fracture" in Vol. 6 of the Handbuch der Physik (1958), Irwin [5] gave a comprehensive account of the various aspects of fracture theory.

This survey article consists of five sections dealing with: 1) fracture resistance in fluids; 2) stress-strain relations in various types of fracture; 3) formation and propagation of cracks; 4) the stress field, velocity, and branching of propagating cracks; and 5) the effect of size on fracture. The subject of the present survey is covered in the second section of Irwin's article, which contains the basic results of quasi-brittle fracture theory. Irwin describes the principal stages in the development of the theory and then points out that the stress state near the crack edge can generally be represented as a superposition of three basic types of stresses: the transverse (rupture or opening) stress related to σ_y, and the two shearing stresses related to τ_{xy}, τ_{yz} (Fig. 12).*

The splitting-type stress state in the case of a planar strain state is described by the formulas

$$F = \operatorname{Re} \bar{\bar{Z}} + y \operatorname{Im} \bar{Z}, \quad \bar{Z} = K_I \sqrt{2z}, \quad z = x + iy$$

$$\sigma_x = K_I \frac{\cos(\theta/2)}{\sqrt{2r}} \left\{1 - \sin\frac{\theta}{2} \sin\frac{3\theta}{2}\right\}$$

$$\sigma_y = K_I \frac{\cos(\theta/2)}{\sqrt{2r}} \left\{1 + \sin\frac{\theta}{2} \sin\frac{3\theta}{2}\right\},$$

*Fig. 12 is taken from Paris and Sih [1]; see also Donaldson and Anderson [1].

$$\sigma_z = \nu(\sigma_x + \sigma_y) = 2\nu K_{\mathrm{I}} \frac{\cos(\theta/2)}{\sqrt{2r}}$$

$$\tau_{xy} = K_{\mathrm{I}} \frac{\cos(\theta/2)}{\sqrt{2r}} \sin\frac{\theta}{2} \cos\frac{3\theta}{2}, \quad \tau_{yz} = \tau_{xz} = 0$$

$$u = \frac{K_{\mathrm{I}}\sqrt{2r}}{2G} \cos\frac{\theta}{2}\left\{1 + 2\nu + \sin^2\frac{\theta}{2}\right\}$$

$$v = \frac{K_{\mathrm{I}}\sqrt{2r}}{2G} \sin\frac{\theta}{2}\left\{2(1-\nu) - \cos^2\frac{\theta}{2}\right\}, \quad w = 0. \quad (1.53)$$

Here F is the stress function and K_{I} is the stress intensity coefficient related to the quantity $\mathcal{G}_{\mathrm{I}}$ by the expression

$$K_{\mathrm{I}} = \frac{2C\mathcal{G}_{\mathrm{I}}}{1-\nu} = \frac{E\mathcal{G}}{\pi(1-\nu^2)}. \quad (1.54)$$

In the case of a planar stress state we have

$$\sigma_z = 0, \; K_{\mathrm{I}}^2 = E\mathcal{G}_{\mathrm{I}}/\pi. \quad (1.55)$$

For the shearing-type stress state due to τ_{xy} we have

$$F = -y \operatorname{Re}\bar{Z}, \quad \bar{Z} = K_{\mathrm{II}}\sqrt{2z}$$

$$\sigma_x = -\frac{K_{\mathrm{II}}}{\sqrt{2r}} \sin\frac{\theta}{2}\left\{2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right\}$$

$$\sigma_y = \frac{K_{\mathrm{II}}}{\sqrt{2r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}, \quad \sigma_z = -2\nu\frac{K_{\mathrm{II}}}{\sqrt{r}}\sin\frac{\theta}{2}$$

$$\tau_{xy} = \frac{K_{\mathrm{II}}}{\sqrt{2r}} \cos\frac{\theta}{2}\left\{1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right\}, \quad \tau_{xy} = \tau_{xz} = 0$$

$$u = \frac{K_{\mathrm{II}}\sqrt{2r}}{2G} \sin\frac{\theta}{2}\left\{2(1-\nu) - \cos^2\frac{\theta}{2}\right\}$$

$$v = -\frac{K_{\mathrm{II}}\sqrt{2r}}{2G} \cos\frac{\theta}{2}\left\{(1-2\nu) - \sin^2\frac{\theta}{2}\right\}, \; w = 0. \quad (1.56)$$

Here K_{II} is the corresponding stress intensity coefficient related to $\mathcal{G}_{\mathrm{II}}$ (i.e. the force producing propagation of the crack in the given slip) or to the shear modulus during propagation of the crack, by the expression

$$K_{\mathrm{II}}^2 = \frac{2G\mathcal{G}_{\mathrm{II}}}{\pi(1-\nu)}. \quad (1.57)$$

Finally for the type of shear stress state produced by the stress τ_{zy} we have

$$u = v = 0, \quad Gw = K_{\mathrm{III}}\operatorname{Im}\;(\sqrt{2z}) = K_{\mathrm{III}}\sqrt{2r}\sin\frac{\theta}{2}$$

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$$

$$\tau_{xy} = \frac{K_{\mathrm{III}}}{\sqrt{2r}}\cos\frac{\theta}{2}, \quad \tau_{xz} = -\frac{K_{\mathrm{III}}}{\sqrt{2r}}\sin\frac{\theta}{2} \quad (1.58)$$

Here K_{III} and $\mathcal{G}_{\mathrm{III}}$ are defined by analogy with the above, and

$$K_{\mathrm{III}}^2 = \frac{2G\mathcal{G}_{\mathrm{III}}}{\pi} \quad (1.59)$$

The three quantities $\mathcal{G}_{\mathrm{I}}$, $\mathcal{G}_{\mathrm{II}}$, $\mathcal{G}_{\mathrm{III}}$ characterize the rates of conversion into other forms of energy as the crack propagates from the surrounding elastic strain field.

Provided they are not accompanied by crack development of the first type, the shearing processes associated with $\mathcal{G}_{\mathrm{II}}$, $\mathcal{G}_{\mathrm{III}}$ do not constitute crack-like strains in the usual sense. Solid bodies (e.g. ceramics) can be very sensitive to local shear. However, localization of the plastic zone near the crack tip in shear does not usually occur in metals, and (notes Irwin) the attention of researchers in the field of quasi-brittle fracture has been concentrated on fractures involving cracks of the cleavage type.

Irwin shows that the solutions of stress concentration problems can be used to find an expression for the stress intensity coefficient

$$K = \lim_{\rho\to 0}(1/2\sigma_{\max}\sqrt{\rho}) \quad (1.60)$$

where $\sigma_{\max}$ is the maximum stress and ρ is the radius at the notch vertex.

He then discusses problems involved in the experimental determination of $\mathcal{G}$. Let us consider a stretched plate with a central crack (Fig. 13).

Such a specimen can be produced by sawing or milling out a narrow hole perpendicular to the direction of tension and by applying the cleaving forces in such a way as to produce a single crack tip at the edge of each cutout. Let us now assume that the crack lengthens by the amount δx. Let the process be slow enough to allow us to neglect the kinetic energy. We denote the corresponding elongation of the plate by δl. The elastic part of the elongation is l_e, and the plastic part l_p. Thus,

$$l = l_e + l_p, \quad \delta l = \delta l_e + \delta l_p. \quad (1.61)$$

Here $F = Ml_e$, where M is the spring constant of the specimen which depends on the configuration of the body (including the size of the crack), and is a decreasing function of the crack length.

The strain energy of the plate is given by

$$U = 1/_2\, Fl_e. \quad (1.62)$$

Hence,

$$\delta U = F\delta l_e - 1/_2\, F^2\delta\,(1/M). \quad (1.63)$$

Let us consider the equation

$$F\delta l = \delta U + \delta W \quad (1.64)$$

which implies that

$$\delta W = F\delta l_p + 1/_2 F^2\delta\,(1/M). \quad (1.65)$$

If we assume that the distance between the fastenings if fixed, then as the crack propagates we have

$$\frac{dW}{dx}\,\delta x = -\,\delta U_1 = F\delta l_p + 1/_2 F^2\frac{d}{dx}\left(\frac{1}{M}\right)\delta x \quad (1.66)$$

where $(-\delta U_1)$ is the strain energy loss in the case of a stretched specimen of constant length. If the plastic elongations are small, then (1.66) implies that

$$\mathcal{G} = \frac{1}{2}F^2\frac{d}{dx}\left(\frac{1}{M}\right). \quad (1.67)$$

Expression (1.67) does not contain terms containing the force increment dF, so that the quantity $\mathcal{G}$ does not depend on the method of loading (e.g. on whether the crack propagates under constant forces or with constant elongation). Relation (1.67) enables us to determine the constant $\mathcal{G}_c$ experimentally.

At the end of the section Irwin cites several experimentally determined constants $\mathcal{G}_c$ for various materials.

He also cites certain new solutions of crack problems obtained by the method of Westergaard. Several collinear planar cracks under an omnilateral tensile stress p at infinity are described by the Westergaard function

$$Z = p\prod_{i=1}^{N}\left\{1 - \frac{a_i}{(z-b_i)^2}\right\}^{-1/2} \quad (1.68)$$

where b_i is a sequence of increasing real numbers and where the quantities a_i are positive

$$b_i + a_i + a_{i+1} < b_{i+1}. \quad (1.69)$$

In this case we have N free-surface cracks

$$|x - b_i| < a_i. \tag{1.70}$$

The stress intensity parameter at the right-hand end of the j-th crack is of the form

$$K_j = p\sqrt{a_j}\prod_{i\neq j}^{N}\left\{1 - \frac{a_i}{(a_j + b_j - b_i)^2}\right\}^{-1/2}. \tag{1.71}$$

In the case of the collinear cracks considered by Westergaard (see the second formula of (1.7)) the stress intensity coefficient turns out to be

$$K = p\left(\frac{l}{\pi}\operatorname{tg}\frac{\pi a}{l}\right)^{1/2}. \tag{1.72}$$

The stress intensity coefficient in the neighborhood of a point in case (1.38) is given by

$$K = \frac{P}{\pi\sqrt{a}}\sqrt{\frac{a+b}{a-b}}. \tag{1.73}$$

Since the quantities Z and K are additive for stress fields, the solution of the problem for any pressure distribution along the crack edges can be obtained from (1.38), (1.73) by setting P = p(b) db and integrating these relations over b.

Though we are unable to describe in detail the rich contents of the other sections of Irwin's article, we must consider his analysis of crack edge structure (Sec. 4, Subsec. II, pp. 557ff.).

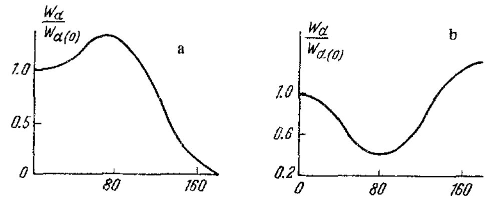

Fig. 7

Analysis of equations (1.53) for an open crack shows that the maximum tensile stress does not lie in the crack plane xz (Fig. 12). The angle θ at which the maximum tensile stress is achieved for a small fixed distance r from the crack tip is $\pm\pi/3$ (Fig. 6). Hence, the maximum fracture near the crack tip can occur at $\theta = \pm\pi/3$, where the components of the shearing stresses are sufficiently large, and where the tensile stresses are maximum. When the new elementary fractures forming away from the idealized site of the principal crack reach a sufficient size, the small but gradually developing defects join them to the principal crack, imparting a complex structure to the edge of the latter. The degree of roughness of the fracture surfaces depends partly on the yield of the material which hinders the formation of local cavities, etc. Owing to the nature of formation of stress fields and their character, it is possible to avoid rough crack surfaces only in exceptional cases, e.g. in the splitting of brittle single crystals. In general the character of crack formation depends on the yield of the material, the directions of its weakening, the nature of the local defects, and the stress fields. For example, in the case of low-carbon steel at room temperature the chief contribution to the formation of small fracture elements is that of the maximum-shearing-stress planes even if the principal crack lies in a plane normal to the maximum normal stress of the overall stress field (Parker, 1957). The chief role in the same material at low temperatures is played by cleavage cracks; the fracture surface contains many small crack formation elements in planes almost parallel to the principal crack.

Irwin places his main emphasis on problems of unstable rapid fracture. In Sec. 5, Subsec. 15, pp. 586ff of his article he discusses several problems concerning the stable (self-arresting) process of crack propagation.

Problems of experimental and theoretical determination of the fracture constants $\mathcal{G}_c$ for rotor-type structures were investigated by Winne and Wundt [1].

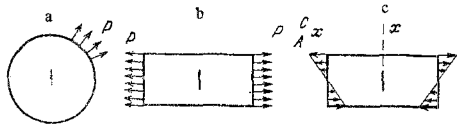

Fig. 8

These authors considered disks with an inner hole weakened by crack-simulating grooves. They also conducted experiments to determine the $\mathcal{G}_c$ of scored specimens in bending, concluding that the quantity $\mathcal{G}_c$ is a material constant. The Winne-Wundt paper provoked a lively discussion by specialists (Felbeck, Irwin, Peterson, Robinson, and Wells).

Experimental determination of the constants $\mathcal{G}_c$ is also the main subject of a paper by Irwin, Kies, and Smith [1]. In their abstract of the paper the authors note that determination of the quantity $\mathcal{G}_c$ together with analysis of the stresses and of the initial crack length is of great importance in analyzing the fracture of jet aircraft fuselages, turbogenerator engines, jet engines, etc.

They describe various experimental methods for determining the constant $\mathcal{G}_c$, namely tests involving the bending of scored specimens, stretching of circular scored specimens, and stretching of centrally scored flat sheets. They discuss the differences between the types of fracture occurring with planar straining and in a planar stress state. In planar straining fracture begins at the center of the scored incision (along the crack edge inside the material); in a planar stress state the shearing fracture begins at the plate surface.

Attainment of the critical value $\mathcal{G}_c$ by the quantity $\mathcal{G}$ is associated by the authors with the onset of rapid unstable crack growth. They attribute this fact to the character of the experiments considered, in which the initiated crack growth led to fracture of the specimen. However, they also describe an experiment on the opening of cracks of length $2a$ in a plate of width l by centrally applied forces P (Fig. 14). Making use of solution (1.40), the authors obtained an expression for of the form

$$\mathcal{G} = \frac{2P^2}{El\sin\left(\frac{2\pi a}{l}\right)}. \tag{1.74}$$

Crack extension in a fixed loading state in such experiments has the effect of reducing the stresses at the crack tip and the crack extension process is stable (self-arresting).

The authors note that the constant $\mathcal{G}_c$ determined through experiments on stable (equilibrium) cracks turns out to be somewhat smaller than its value obtained by experiments involving stable crack growth.

This fact has to do with the dependence of the material properties on the straining rate: the more rapid the crack growth, the greater the resistance.

The authors suggest that $\mathcal{G}_c$ or K_c in this case be determined by way of the relationship between the applied load and the crack length, in terms of (1.74).

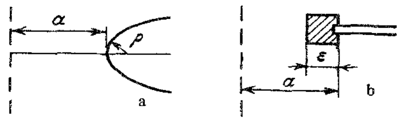

Fig. 9

They also discuss the effects of temperature, specimen thickness, and dynamic load on fracture. In the final section of the paper on "Applications of Fracture Resistance Analysis" the authors discuss the prevention of cracks in such structures as pressurized aircraft cabins. They describe a method for computi g the appropriate crack arrest conditions.

In his detailed paper [7] Irwin again presents the principal concepts of quasi-brittle fracture mechanics. He cites several new expressions for the stress concentration coefficients. The paper includes a detailed discussion of experimental data as does his article [5].

In 1958 Bueckner published a paper [1] concerning the changes in the elastic energy of a body during crack development.

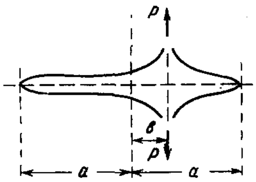

Fig. 10

Following Bueckner, let us consider a body of volume v bounded by the surface $S = S_p + S_u$ (Fig. 15). We denote the body forces by X_i; the forces p_i are given on the portion of the surface S_p, and the displacements u_{i0} are given on S_u. The initial stress-strain state of the body with a crack prior to its development is characterized by the tensors σ_{ij}, e_{ij} and by the displacement vector u_i.

Let us assume that the side surfaces a_1, a_2 of the crack separate under the load and that they are free of surface forces. Let us consider the virtual extension of the crack which results in the formation of its additional side surfaces a_1', a_2' indicated by dashed in Fig. 15. The mass and surface forces are assumed constant. The stress-strain state of the body with the altered crack (the varied state) is characterized by the components σ_{ij}', e_{ij}', and u_i', respectively.

From now on we can interpret the initial state as the state of a body with an extended crack. The completely defined forces p_i^* must then act along a_1', these forces hold the crack in the closed state over this segment (the fracture $[u_i]$ over the segment of length $|a_1'|$ in the initial state is equal to zero). It is clear that the vector p_i^* assumes the opposite direction at the opposite edge of the crack.

The change in the work done by the external forces during propagation of the crack is of the form

$$W_e = \int_V X_i (u_i' - u_i)\, dv + \int_{S_p} p_i (u_i' - u_i)\, dS. \qquad (1.75)$$

The increment in the work of the internal forces can be expressed as

$$W_i = U - U' \qquad (1.76)$$

where the strain energies of the initial and varied states are given by

$$U = \frac{1}{2}\int_V \sigma_{ij} e_{ij}\, dV, \quad U' = \frac{1}{2}\int_V \sigma_{ij}' e_{ij}'\, dV. \qquad (1.77)$$

Bueckner writes the condition of crack propagation in the form

$$W = W_e + W_i > \mathscr{G}_c\, |a_1'| \qquad (1.78)$$

where W is the total work increment and a_1' is the crack area increment.

Let us introduce the following expression for the mixed energy of the body:

$$U_m = \frac{1}{2}\int_V \sigma_{ij}' e_{ij}\, dv = \frac{1}{2}\int_V \sigma_{ij} e_{ij}'\, dV. \qquad (1.79)$$

Expressions (1.76), (1.77), and (1.79) imply that

$$-W_i = U' - U = \frac{1}{2}\int_V (\sigma_{ij}' + \sigma_{ij})(e_{ij}' - e_{ij})\, dV. \qquad (1.80)$$

Let us consider two ancillary states: the "sum" state with the stresses $\overset{+}{\sigma}_{ij} = \sigma_{ij}' + \sigma_{ij}$, the strains $\overset{+}{e}_{ij} = e_{ij}' + e_{ij}$, and the displacements $u_i = u_i' + u_i$, and the "differencd" state with the respective components $\sigma_{ij} = \sigma_{ij}' - \sigma_{ij}^- = e_{ij} - e_{ij}^-$, $u_i^- = u_i' - u_i$.

In accordance with the above interpretation of the initial state the sum and difference states can be expressed as follows.

The sum state corresponds to the stressed and strained body V with an extended crack under the body forces $2X_i$, the forces $2p_i$ at S_p, the displacements $2u_{i0}$ at S_u, zero surface forces at a_1, a_2, and the surfaces forces p_i^* along a_1', a_2'. Similarly, the difference state corresponds to the absence of body forces in the body, to the absence of forces and displacements at S_p and S_u, and to the absence of surface forces at a_1, a_2. It is determined by the surface forces p_i^* at a_1', a_2' only.

Now, by virtue of the Betti theorem we can convert from the internal to the external forces and rewrite integral (1.80) in the form

$$-W_i = U' - U = \frac{1}{2}\int_V \sigma_{ij}^+ e_{ij}^-\, dV =$$

$$= \int_V X_i u_i^-\, dV + \int_{S_p} p_i u_i^-\, dS + \frac{1}{2}\int_{a_1'+a_2'} p_i^* u_i^-\, dS. \qquad (1.81)$$

From (1.81) and (1.75) we obtain

$$W = W_e + W_i = -\frac{1}{2}\int_{a_1'+a_2'} p_i^* u_i^-\, dS \qquad (1.82)$$

Applying the Clayperon theorem and converting from the external to the internal forces, we transform (1.82) into

$$W = -\frac{1}{2}\int_{a_1'+a_2'} p_i^* u_i^-\, dS = \frac{1}{2}\int_V \sigma_{ij}^- e_{ij}^-\, dV = U^{(-)} \qquad (1.83)$$

where $U^{(-)}$ is the strain energy of the difference state.

The energy $U^{(-)}$ of the difference state is always nonnegative. Analysis of relations (1.78) and (1.82) indicates that the required change in energy W is equal to the work done by the forces p_i^* in closing the crack over the corresponding displacements. Irwin used this as a hypothesis in deriving formula (1.48). Bueckner justified this procedure.

We infer from (1.78), (1.82), and (1.83) that the condition of crack propagation can be written as

$$\lim_{a_1' \to 0} \frac{U^{(-)}}{|a_1'|} = \frac{\partial U^{(-)}}{\partial a} \geqslant \mathscr{G}_c \qquad (1.84)$$

where $\partial U^{(-)}/\partial a$ is the rate of liberation of elastic energy.

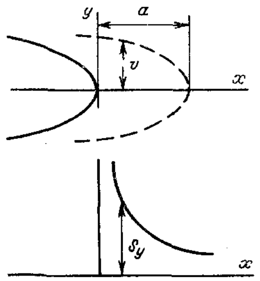

Fig. 11

Bueckner noted that since crack propagation is affected only by the energy $U^{(-)}$ of the difference state, nothing is altered by subtracting the same stress-strain state from the initial and varied states. This state can be taken in the form of the components $\overset{\circ}{\sigma}_{ij}$, $\overset{\circ}{e}_{ij}$, $\overset{\circ}{u}_{ij}$ characterizing the stress-strain state of the uncracked body V acted on by the same body forces and boundary conditions at S. The stress-strain state $\sigma_{ij} - \overset{\circ}{\sigma}_{ij}$, $e_{ij} - \overset{\circ}{e}_{ij}$, $u_i - u_i^\circ$ corresponds to the state of the body V in the absence of body forces with zero boundary conditions at S. The crack is acted on by the forces p_i^0 arising in the uncracked body at the surface corresponding to the crack surface. We can therefore assume that extension of the crack depends solely on the forces p_i^0 applied to the banks of the crack.

Bueckner then considers two special cases. Let us assume that $S_u = 0$, i.e., that the forces p_i are defined over the entire side surface S of the body. In this case integral (1.81) can be transformed into

$$U' - U = \frac{1}{2}\int_V \sigma_{ij}^+ e_{ij}^- \, dV =$$

$$= \frac{1}{2}\int_V \sigma_{ij}^- e_{ij}^+ \, dV = -\frac{1}{2}\int_{a_1'+a_2'} p_i u_i^* \, dS \qquad (1.85)$$

with the aid of the Betti theorem.

Noting that

$$\int_{a_1'+a_2'} p_i^* u_i \, dS = 0 \qquad (1.86)$$

since the quantity u_i is the same at the surfaces a_1, and since p_i^* assumes the opposite sign, we obtain

$$\int_{a_1'+a_2'} p_i^* u_i^- \, dS = \int_{a_1'+a_2'} p_i^* u_i^+ \, dS. \qquad (1.87)$$

Comparing (1.82), (1.85), and (1.87), we find that

$$W_e = 2U^{(-)}, \; W_i = U - U' = -U^{(-)} \; W = U^{(-)}. \qquad (1.88)$$

Thus, the strain energy diminishes under a constant load, but the applied forces supply double the strain energy. It is the difference between the two energies which is expended on crack extension.

Let us now consider crack extension in the case of fixed boundaries. We assume here that the varied state experienced the same displacements as the initial state along S_p and the same fixed displacements u_{i0} along S_u. In this case the displacements u_i^- of the difference state are equal to zero along S. The expression for W_e then represents the virtual work done by the body forces alone. If there are no body forces, then

$$W = U^{(-)} = U - U'. \qquad (1.89)$$

The strain energy diminishes with crack extension, and this decrease contributes to the extension process.

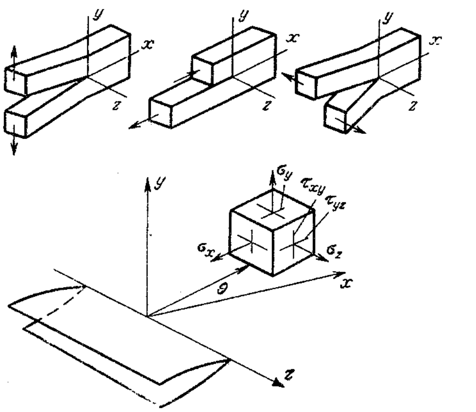

Fig. 12

Bueckner notes that the energy $U^{(-)}$ of the difference field in the case of fixed boundaries is smaller than the energy of the difference field in the case of fixed loads. This difference tends to zero for infinitely small crack extension, however. He considers a one-dimensional elastic system as an example. His argument is entirely similar to that used by Irwin [5] in connection with relations (1.61)-(1.67).

Bueckner derives Irwin's formula (1.46) in the case of crack development under a constant load. Similar arguments can be applied in the case of fixed boundaries. The crack propagation criterion becomes

$$\mathscr{G} \geqslant \mathscr{G}_c. \qquad (1.90)$$

He considers a cracked rotating disk as an example. Irwin took part in the discussion of his results.

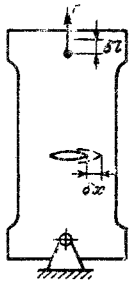

Fig. 13

By 1957/58 the studies of Irwin, Orowan, Bueckner, et al. contained the final formulation of quasi-brittle fracture theory for the principal cases of straining. Ample experimental evidence in support of the theoretical findings was adduced in due course.

§3. Other approaches to the theory of quasi-brittle fracture. Working outside the framework of Griffith's studies, Westergaard [1] (1933) and [2] (1939) proposed a force approach to the theory of cracks in brittle materials such as concrete.

In his first paper [1] Westergaard investigates ferro concrete beams with cracks in the concrete. He proceeds on the basis of the condition that the tensile stresses cannot occur at the tip of a crack in a concrete beam compressed by the force P (Fig. 16a), and that the crack in this case must lengthen until such time as the stress curve assumes the form shown in Fig. 16b. (We take this opportunity to note the fallacy of Barenblatt's statement on p. 12 of his paper [9]: "Westergaard did not relate the condition of finite stresses to the determination of crack length, which he assumed to be given".)

Westergaard's analysis is based on the solution of the problem of elasticity theory in the absence of a tensile stress concentration near the crack tip

$$\sigma_r = \frac{K\sqrt{r}}{4}(-\sin 2.5\theta - \sin 0.5\theta)$$

$$\sigma_\theta = \frac{K\sqrt{r}}{4}(\sin 2.5\theta - 5\sin 0.5\theta)$$

$$\tau_{r\theta} = \frac{K\sqrt{r}}{4}(-\cos 2.5\theta + \cos 0.5\theta) \qquad (1.91)$$

where K is a constant which depends on the magnitude of the compressive forces. The coordinate system is shown in Fig. 4b.

His solution is approximate since the boundary conditions are not satisfied exactly and the effect of the reinforcement is allowed for approximately. However, these conditions do not affect the character of stress and displacement distribution near the crack tip.

The solution of elasticity theory yields a relation giving the crack width in the form

$$v = \frac{8K}{3E} r^{1.5}\left(1 - \frac{2}{5a}\right). \qquad (1.92)$$

Equation (1.92) implies that the merging of the crack banks at the tip must be smooth, i.e., that

$$\left.\frac{dv}{dr}\right|_{r=0} = 0. \qquad (1.93)$$

Having allowed for the approximate character of his expressions, Westergaard [1] arrived at a wedge-shaped crack opening,

$$\left.\frac{dv}{dr}\right|_{r=0} = \frac{8K\sqrt{a}}{3E}. \qquad (1.94)$$

In his paper [2] mentioned in Sec. 1 above, Westergaard gives the exact solution for the opening of a crack by concentrated forces under the condition of finite tensile stresses at the crack tip.

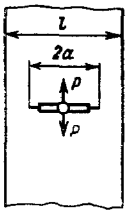

Fig. 14

Notions concerning the smooth merging of crack edges were developed by Elliot [1] (1947). Rebinder [1] (1947) noted the wedge shape of a crack at its tips. Frenkel [1] (1952) attempted to refine the Griffith theory to cover cracks with smoothly merging edges.

Zheltov and Khristianovich [1] (1955) investigated the problem of an infinitely elastic body weakened by a straight-line crack. They assumed that a plane is compressed by a uniform pressure at infinity and that a uniform pressure is applied over a finite segment of the crack banks (smaller than the length of the crack). These authors formulated the finite-stress condition independently of Westergaard [1, 2]: "the stresses at the crack tips in rock must be finite; the crack could not terminate otherwise". They used this condition to determine the dependence of crack length on the applied forces. In subsequent studies this hypothesis was used in dealing with certain problems of crack propagation in elastic materials with special reference to mining and oil drilling problems (Barenblatt and Khristianovich [1] and Zheltov [1, 2]).

Barenblatt's paper [1] was mentioned in Sec. 1. He determined the relationship between crack length and load on the basis of the hypothesis of finite stresses at the crack tips. Using the Sneddon solution [2], he also considered certain particular solutions for a three-dimensional circular crack acted on by an axisymmetric load.

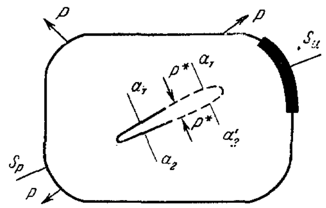

Fig. 15

The aforementioned solutions of Westergaard [1, 2], Zheltov and Khristianovich [1], and Barenblatt [1, 2] correspond to the case T = 0 from the Griffith standpoint and to the case $\mathcal{G}_c = 0$ from the standpoint of Irwin and Orowan.

Rzhanitsyn [1] (1956) proceeded (as did Frenkel [1]) from the notion of a sharp-tipped circular crack. Noting certain inaccuracies in Frenkel's analysis, he introduced a small tip zone α of crack edge interaction forces. Figure 17a shows the crack tip, and Fig. 17b the curve of these crack edge interaction forces. Here σ_0 is the tensile stress due to the external load, σ_* is the theoretical tensile strength, R is the area under the interaction force curve (or, as Rzhanitsyn calls it, the curve of "additional stresses"), δ is the distance between the banks of the layer containing the crack, and δ_* is the distance corresponding to the maximum R or σ_*. The quantity R does not depend on σ_0 and is given by the formula

$$R = \int_0^\alpha \sigma(\delta)\,dx. \tag{1.95}$$

Rzhanitsyn then carried out an energy analysis of crack extension and estimated critical crack sizes. He also developed a method for calculating the critical crack radius in the case of a given dilatation.

In his later papers Barenblatt [3, 4] (1959) followed Rzhanitsyn [1] in introducing a small zone near the crack edge characterized by the action of coupling forces. In [4] he considered the case of a space weakened by a penny-shaped crack under an axisymmetric load.

On the basis of the smooth-merging condition and Sneddon's solution [2], he obtained the following crack propagation condition:

$$\int_0^a \frac{rp(r)\,dr}{\sqrt{a^2-r^2}} = K\sqrt{\frac{a}{g}}, \quad K = \text{const} \tag{1.96}$$

where a is the radius of the crack, p(r) is the load, and K is the coupling modulus.

In [5] he considered the planar problem. The expression analogous to (1.96) turned out to be

$$\int_0^a \frac{p(x)\,dx}{\sqrt{a^2-x^2}} = \frac{K}{\sqrt{2a}}, \quad K = \text{const}. \tag{1.97}$$

The same paper also contains the following general rule: "the crack tips are determined by the condition that the stresses acting there are computed without allowance for the coupling forces go to infinity according to the law

$$K / \pi\sqrt{s}\text{''}. \tag{1.98}$$

This rule makes it possible to exclude the coupling forces as such from consideration.

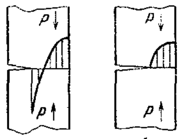

Fig. 16

We note that the quantity s characterizes the distance from the crack tip and coincides with the r in (1.53).

The above quotation implies that Barenblatt's constructions led him to the crack development criteria of the Griffith-Irwin theory. (We note in this connection that some authors have displayed insufficient acquaintance with the history of the problem. Thus, in his abstract of Sih's paper [6] (RZh Mekhanika, no. 2, 1966, 2V 363) R. A. Salganik writes: "The author errs in stating that the approach based on intensity coefficients was originated by Irwin. The theory of cracks based on such notions was developed by G. I. Barenblatt (PMTF, no. 4, 1961)")

In [6] he considered the relationship between the force and energy approaches in quasi-brittle fracture theory. He obtained expressions for the rate of liberation of elastic energy for a space containing a single crack in the case of the axisymmetric and planar problems,

$$\frac{\partial W}{\partial a} = \frac{8(1-\nu^2)}{E}\left\{\int_0^a \frac{rp(r)\,dr}{\sqrt{a^2-r^2}}\right\}^2,$$

$$\frac{\partial W}{\partial a} = \frac{8(1-\nu^2)\,a}{E}\left\{\int_0^a \frac{p(x)\,dx}{\sqrt{a^2-x^2}}\right\} \tag{1.99}$$

where W is the elastic energy.

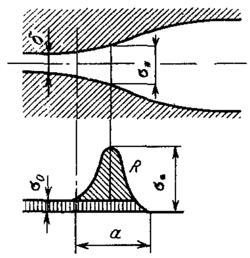

Fig. 17

Comparison of the two equations of (1.99) and of (1.96) with (1.97) indicates the equivalence of the force and energy approaches. (The establishment of this equivalence and the development of the force method was one of the principal achievements of the Griffith-Irwin theory. It was accomplished in the general case in the aforementioned studies of Irwin, Bueckner, et al.)

In 1960 there appeared Barenblatt's paper [7] concerning the finiteness condition in continuum mechanics. Barenblatt used the finiteness condition as the basis of his crack theory, whose concepts are set forth in papers [8-11]. The results are discussed in papers by Ivlev [1] and Cherepanov [11].

In [9] (1964) Barenblatt introduced the universal coupling coefficient function (see also Sih, Paris, and Erdogan [1], (1962)) and considered the propagation of equilibrium cracks, the construction of the limiting surface of lead parameters λ_i, and modeling in the case of brittle fracture.

In 1964 Keer [1] applied the notions of coupling forces and smooth merging of crack banks. Working within the framework of classical elasticity theory, he determined the coupling stresses and the crack zone characterized by the action of coupling stresses. He also considered an axisymmetric crack in the cases of homogeneous tension and shear, and also the analogous two-dimensional problem with homogeneous tension. In their monograph [1] Landau and Lifshits (1965) present the theory of a smoothly merging crack in an elastic medium using the mathematical apparatus of dislocation theory. They note that "in its literal from the above theory... is, in fact, valid for ideall brittle bodies, i.e. for bodies which retain linear elasticity until fracture (e.g. glass, fused quartz)."

The stress state near the tip of a smoothly merging (parallel) crack was also considered by Baker [1]. Problems concerning the fracture criterion for the development of a crack with coupling forces acting at its tips were discussed by Willis [1].

Let us now make some remarks concerning studies based on the finite-stress hypothesis. By (1.26), the change in energy associated with the formation of a crack is given by

$$\delta W = \delta \Pi. \qquad (1.100)$$

The work expended on the formation of the new free crack surfaces is different from zero, so that $\delta\pi \neq 0$. This means that the change in the elastic energy of the body during crack formation is always different from zero, i.e. that $\delta W \neq 0$. By (1.35), the quantity $\mathcal{G}$ differs from zero for $\delta W \neq 0$, so that the finite-stress hypothesis can be used as an assumption based on the possibility of neglecting δW in those cases where the material constant $\mathcal{G}_c$ is small even though, strictly speaking, $\mathcal{G}_c \neq \neq 0$. The authors of the theory of quasi-brittle fracture assumed that the material in the tip zone is not subject to the laws of linear elasticity theory and at the force condition can be formulated on the basis of the relationship between the stress intensity coefficients and the rate of elastic energy liberation in the body. According to the estimates of Irwin et al. the contribution of the tip zone to the general expression for the rate of energy liberation in quasi-brittle fracture is negligible.

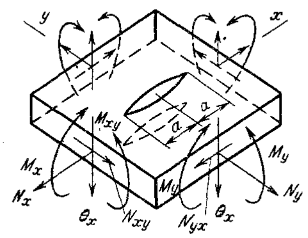

Fig. 18

The above models based on the notion of smoothly merging crack banks are based essentially on the preservation of the laws of linear elasticity theory in the tip zone. The coupling forces are assumed to produce displacements which conform to the linear elasticity laws. This view is not valid in the desciption of quasi-brittle fracture.

We recall that, by definition, brittle fracture is accompanied by the formation of a small plastic zone near the crack tips.

§4. Further developments in quasi-brittle fracture theory. After the appearance of the fundamental studies of Griffith, Irwin, and other researchers the theory of quasi-brittle fracture continued to develop rapidly largely through the expansion of its range of applications.

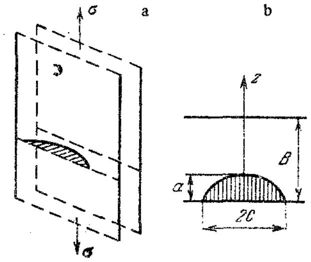

Fig. 19

An integral interpretation of the Griffith-Irwin theory was developed by Sanders [1] (1960). Let us consider a plate containing a crack of length $L(\alpha)$, where α is a parameter which increases with increasing L. Let the stress, strain, and displacement components σ_{ij}, e_{ij}, u_i be known functions of the coordinates x, y, of the time t, and of the parameter α. Further, let C be a closed curve surrounding the crack. According to the Griffith-Irwin theory, virtual changes in the length of the crack entail an energy balance: the rate of work performed by the forces at the contour C is equal to the rate of increase of the strain energy stored inside the contour C in the material plus the rate of energy expenditure on the lengthening of the crack, i.e.

$$\int_C T_i \frac{du_i}{dt} ds = \frac{1}{2} \frac{d}{dt} \int_C T_i u_i ds + \mathcal{G} \frac{dL}{d\alpha} \frac{d\alpha}{dt} \qquad (1.101)$$

where T_i are the forces at the contour and is a constant.

Equation (1.101) can be transformed into

$$I = \frac{1}{2} \int \left(T_i \frac{\partial u_i}{\partial \alpha} - u_i \frac{\partial T_i}{\partial \alpha} \right) dS = \mathcal{G} \frac{dL}{d\alpha}. \qquad (1.102)$$

Relation (1.102) is the Griffith-Irwin fracture criterion expressed in integral form. Relation (1.102) must be independent of the contour C.

Sanders [1] noted that the results remain valid if the contour C is open and begins and ends at the free surfaces of the crack, whose tip it surrounds. He rewrites criterion (1.102) in terms of Muskhelishvili functions [1]. As an example he considers a rectangular plate containing a crack in the case of fixed boundaries and in the case of a constant load. Another of his examples relates to the Westergaard solution (1.7).

Survey [1] by Donaldson and Anderson concerns applications of quasi-brittle fracture theory to aircraft structures. These authors summarize the principal theoretical results accumulated by 1961.

Williams [5] investigated the determination of the singularity of the solution near a crack edge in a bent plate provided by elasticity theory. (Some of Williams' hypotheses are discussed in communication [1] by Redwood and Shepherd.)

The results obtained by Williams [5] were later used by Sih, Paris, and Erdogan [1] for determining the corresponding stress intensity coefficients in bending. The latter paper also contains a discussion of the destruction of a part of a crack-weakened sheet structure loaded as in Fig. 18. The stress intensity coefficients due to the action of the forces in the plane of the plate are given by

$$K_I = \frac{N_y}{h} a^{1/2}, \quad K_{II} = \frac{N_{xy}}{h} a^{1/2} \qquad (1.103)$$

where h is the thickness of the plate.

The intensity coefficients in bending are of the form

$$k_1 = \frac{6M_y}{h^2} a^{1/2} + \frac{8\nu Q a^{3/2}}{h^2}, \quad k_2 = \frac{6M_{xy}}{h^2} a^{1/2} + \frac{8Q a^{3/2}}{h^2}. \qquad (1.104)$$

Extension of the Griffith-Irwin theory to such problems leads to the following conclusion: unstable crack development begins upon attainment of a certain limiting value

$$f(K_I, K_{II}, k_1, k_2) = 0 \qquad (1.105)$$

by a certain combination of stress intensity coefficients.

However, this statement does not take into account the interaction of the crack surfaces. In the case of bending, one side of the crack propagates faster than the opposite side. This fact is also ignored in relation (1.105). The authors note that the introduction of relation (1.105) is a step towards the more general use of the Griffith-Irwin theory, although any practical investigation must be accompanied by verification of the implications of this procedure. In conclusion they note that the chief contribution of their study is to point the way to the more general use of the Griffith-Irwin fracture theory.

In his paper [8] Irwin carries out an approximate analysis of intensity coefficients for a blind crack in a plate (Fig. 19). He assumes for simplicity that the crack is a semi-ellipse in plan whose boundary is described by the equation

$$\frac{x_1^2}{a^2} + \frac{z_1^2}{c^2} = 1, \quad a < c. \qquad (1.106)$$

The study of a planar elliptical crack carried out by Green and Sneddon [1] showed that the tension σ normal to the crack makes it ellipsoidal in shape. Irwin assumes that the normal displacement of the crack surface is given by

$$\eta = \eta_0 \left(1 - \frac{x^2}{a^2} - \frac{z^2}{c^2}\right)^{1/2} \qquad (1.107)$$

where η_0 is one-half of the maximum distance between the crack boundaries.

Expressing the boundary of the ellipse parametrically in the form $x_1 = a \sin\varphi$ and $z_1 = c\cos\varphi$, Irwin arrives at an expression for the quantity $\mathscr{G}$, namely

$$\mathscr{G} = \frac{\pi(1-\nu^2)\sigma^2}{E\Phi}\left(\frac{a}{c}\right)(a^2\cos^2\varphi + c^2\sin\varphi)^{1/2} \qquad (1.108)$$

where Φ is an elliptical integral,

$$\Phi = \int_0^{\pi/2} \left(\sin^2\varphi + \left(\frac{a}{c}\right)^2 \cos^2\varphi\right)^{1/2} d\varphi. \qquad (1.109)$$

Analysis of relation (1.108) indicates that $\mathscr{G}$ assumes its maximum value when the crack boundary intersects the minor axis of the ellipse. Thus, an elliptical crack must tend to become circular with increasing tension σ (this does not, of course, apply in the case of an anisotropic material).

Irwin discusses the corrections which must be made in applying the above results to blind cracks. His final expression for the intensity coefficient is of the form

$$K = \frac{1.2\sigma^2 a}{\Phi^2 - 0.212(\sigma/\sigma_{ys})^2} \qquad (1.110)$$

where σ_{ys} is the yield strength of the material in uniaxial tension.

He also discusses the corresponding experimental data of Srowley.

Problems of determining the stress intensity coefficients near elliptical holes are also discussed by Panasyuk [4–6, 9, and 12] and by Kassir and Sih [1].

Cherepanov formulated several problems and adduced some considerations concerning crack development and fracture. In [4] he considers crack development during the sinking of an indenter. In [5] he develops a hydrodynamic formulation of certain static problems concerning cracks in solids. In [7] he considers crack propagation in compressed bodies.

Sih [2] discussed the extension of the notion of quasi-brittle fracture theory to the case of thermal stresses. Using the Kosolov-Muskhelishvili formulas, he obtained the following expressions for the stress intensity coefficients:

$$K_I = \frac{2}{\sqrt{a}}\,\mathrm{Re}\,[A], \quad K_{II} = -\frac{2}{\sqrt{a}}\,\mathrm{Im}\,[A] \qquad (1.111)$$

where a is one-half of the crack length and where the constant A occurs in the expressions for the stress functions in the case of a steady two-dimensional thermal stress field,

$$\varphi(z) = A\log z + \varphi_0(z), \ \psi(z) = \bar{A}\log z - \psi_0(z). \qquad (1.112)$$

Other studies include that of Jahsman and Field [1] who discuss the generalization of the energy notions of Griffith for the case where residual (self-balancing) stresses are present in a body.

McClintock and Walsh [1] consider the application of Griffith crack theory to rocks under pressure.

Paris and Sih summarized the state of knowledge gained by 1964/65 in their extensive survey [1] of foreign materials.

Mossakovskii and Rybka [1] (1965) proposed a method for constructing a strength theory based on the notions of material fissility. They assume that the cracks are perpendicular to the surface of the plate, that they propagate in straight lines, and that the surface tension and crack length are material constants.

In his paper [1] (1966) Cotterell analyzes the expansion of stresses in power series (1.18)–(1.20) about the crack tip in the symmetric case $b_i = 0$ (see Williams [3]). He shows that if the first coefficient of the expansion is related to the stress concentration coefficient and characterizes the onset of fracture, then the coefficient of the second term of the expansion represents the stability of the direction of crack development. The coefficient of the third term of the expansion represents the stability of crack propagation, and, finally, the fourth coefficient represents the increase or decrease of the maximum shearing stresses along the line extending from the crack with increasing distance from the crack tip.

Swedlow [1] discussed the possibility of using Griffith's ideas in allowing for the effect of stress component distribution on fracture.

Spencer [1] considered the change in energy in a body due to the formation of a crack. In contrast to Sneddon's case [1], the forces and displacements are applied to the plate contour rather than at the crack surface. Similar problems are dealt with by Sih and Liebowitz in [3].

We must also take note of the general analysis of the energy change associated with crack propagation carried out by Rice [3].

Cherepanov [8] (1967) analyzed the conditions of crack propagation on the basis of general energy considerations. Surrounding the crack tip by a circle of radius R, he obtained, for example, the following energy balance equation for the case of isothermal development of quasistatic crack*:

*Relation (1.101) can be rewritten in the form (1.113). Let us set $\alpha = L$ and direct the x-axis along the crack. Making use of the fact that $\partial()/\partial L = -\partial()/\partial x$ (Cherepanov [8]), we find that relation (1.101) becomes

$$\frac{1}{2}\frac{d}{dx}\int_C T_i u_i ds - \int_C T_i \frac{\partial u_i}{\partial x} ds = \mathscr{G}_c. \qquad (1)$$

The Clayperon theorem and the Gauss-Ostrogradskii formula imply that

$$\frac{1}{2}\frac{d}{dx}\int_C T_i u_i ds = \frac{d}{dx}\iint_D W^\circ dx\,dy = \int_C W^\circ \cos(nx)\,ds \qquad (2)$$

where W^0 is the specific elastic potential equal to the specific internal energy in the case of an elastic body; (nx) is the angle between the normal to the contour C and the x-axis.

Substituting (2) into (1), we obtain relation (1.113).

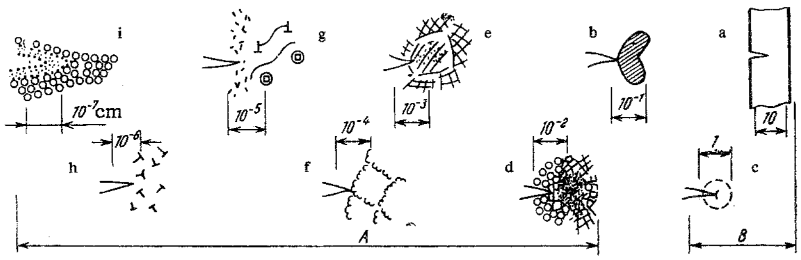

Fig. 20. A) planar strain state; B) planar stress state; a) specimen; b) elastic singularity; c) elasticoplastic state; d) large plastic strains; e) grains, inclusions, cavities; f) fine-grain slip bands; g) fine-grain structure; h) dislocations; i) ion and electron cover.

$$R \int_0^{2\pi} \left[\rho U \cos\theta - (\sigma_x \cos\theta + \tau_{xy} \sin\theta) \frac{\partial u}{\partial x} - (\tau_{xy} \cos\theta + \sigma_y \sin\theta) \frac{\partial v}{\partial x}\right] d\theta = 2T \qquad (1.113)$$

where ρU is the internal energy per unit volume; T is the surface tension (or the surface energy density).

Making use of asymptotic expansions (1.53) and of the equality of the internal energy ρU of an elastic body to its specific elastic potential,

$$\rho U = W^\circ = \frac{1}{2}(1-\nu^2) E^{-1} (\sigma_x + \sigma_y)^2 + (1+\nu) E^{-1} (\tau_{xy} - \sigma_x \sigma_y)$$

Cherepanov arrives at Irwin's formula for the rate of energy liberation.

Ershov and Ivlev [1] (1967) formulate the problem of determining the direction of crack development on the basis of energy considerations.

The theory of quasi-brittle fracture is one of the simplest phenomenological theories of fracture, but its simplicity excludes a high degree of generality. The fracture of various materials under various conditions is by no means always quasi-brittle, and general fracture theroy requires further study and elaboration.

Figure 20 (taken from McClintock and Irwin [1]) shows the range of basic phenomena which can be used as a basis for considering the crack development process. The thickness-to-width ratio of the specimen is $\approx 10^{-2}$. Until such time as crack development has been analyzed on the basis of the atomic notions of solid-state physics it will be necessary to use various model, each of which has its own range of applicability for describing fracture processes. The more perfect the fracture mechanism at the basis of a model, the more reason there is to expect an extension of the range of describable fracture-related phenomena.

The theory of quasi-brittle fracture is in this sense the simplest model; the characteristic scale of the effects which it describes is designated by unity in Fig. 20.

At the same time we note that the various fracture models which entail the analysis of elastic stress fields and the hypothesis of local fracture near the crack tip for each individual type of crack inevitably led to the formulation of the force conditions of the Griffith-Irwin theory.

Fracture theory is receiving much attention abroad. Let us note some of the milestones in its development in recent years.

In 1961 there appeared Sneddon's monograph [3] The Crack Problem in the Mathematical Theory of Elasticity based on a series of lectures presented at North Carolina State College.

In September 1961 Cranfield, England was the scene of an international symposium on crack propagation whose proceedings appeared in two volumes in 1962 (Proceedings of the Crack Propagation Symposium, The College of Aeronautics, Cranfield (1962)).

An international conference on fracture problems was held in Washington in 1962. The papers presented appeared in the volume Fracture of Solids, Proceedings of the International Conference in 1963 (2nd ed., 1965). The Russian translation of the Proceedings published by "Metallurgiya" Press in 1967 omits the section on macroscopic crack theory. The Russian edition does not include the fundamentally important paper by Drucker (which the reader will find translated in the collection "Mekhanika", no. 1, 1964) and seven papers by the following authors: Craggs, McClintock [3], Goodies and Field [1], Marsh (two papers), Williams [7], and Grundfest.

In 1959 the US Department of Defense organized the special E-24 Committee under the auspices of the ASTM (American Society for Testing and Materials) to investigate the strength of structures weakened by cracktype defects. The need for a quantitative approach to the problem of crack defect tolerances began to be felt in the US in connection with recurring structural failures of the early Polaris rockets. In 1964 the Committee held a conference in Chicago. Its proceedings appeared under the title Fracture Toughness Testing and Its Applications (Philadelphis, 1965).

Problems of fracture theory have been discussed at various international and national congresses, conferences, and symposia in the US, Great Britain, Japan, and other countries.*

An international conference on fracture theory was held in Japan in 1965 (see the three-volume Proceedings of the First International Conference on Fracture, held in Sendai, Japan in 1965 (1966)). The majority of the papers presented dealt with elasticoplastic, viscoplastic, fatigue, and other types of fracture.

§5. The relationship between the quasi-brittle fracture and stress concentration theories. The theory of quasi-brittle fracture is directly related to the theory of stress concentrations in elastic media.

If σ_{max} is the maximum stress and ρ is the radius of curvature at the vertex of a groove or hole, and if the groove or hole becomes a slit of zero width as $\rho \to 0$, then the stress intensity coefficient is given by

$$K_I = \lim_{\rho\to 0} \left(\frac{\sqrt{\pi}}{2} \sigma_{max} \rho^{1/2}\right) \qquad (1.114)$$

for cracks of the first type, and by

$$K_{II,III} = \lim_{\rho\to 0} (\sqrt{\pi}\, \sigma_{max} \rho^{1/2}) \qquad (1.115)$$

for cracks of the second and third type.

The solutions of problems of stress concentration theory obtained by Neuber [1], Savin [1], Peterson [1], Isida [1], et al., can be used directly for determining stress intensity coefficients in crack theory.

Following Paris and Sih [1], we can write out the formulas for the stress intensity coefficients in the case of the planar problem. If the Westergaard function near the crack tip is given in the form Z(z),

*We take this opportunity to call attention to the need for more translation into Russian of papers on fracture theory in general and on the theory of quasi-brittle fracture in particular. We still lack Russian translations of the basic foreign materials on quasi-brittle fracture. Not a single paper on quasi-brittle fracture theory has appeared in the Collection: "Mekhanika" in the last decade. Translations of some articles on fracture theory will be found in (J. Appl. Mech.) which has been appearing since 1961 in Russian, and also in other journals in the same series.

where $z = re^{i\theta}$, then

$$K_i = \lim_{|z|\to 0} (2\pi z)^{1/2} Z_i, \quad i = I, II, III. \tag{1.116}$$

The formulas for the intensity coefficients given in the first subsection of the present paper can be obtained from (1.116).

As already noted, the Westergaard formulas are a special case of the Kolosiv-muskhelishvili formulas. The general expression for the biharmonic function Φ can be written as

$$\Phi = \mathrm{Re}\,[\bar{z}\varphi(z) + \chi(z)]. \tag{1.117}$$

The sum of normal stresses becomes

$$\sigma_x + \sigma_y = 4\mathrm{Re}\,[\Phi'(z)] \tag{1.118}$$

where the prime denotes the derivative with respect to z.

Introducing the stress intensity coefficient

$$K = K_I - iK_{II} \tag{1.119}$$

we obtain the following general expression for a crack with its vertex at the point z_1:

$$K = K_I - iK_{II} = 2\sqrt{2\pi}\lim_{z\to z_1}(z - z_1)^{1/2}\Phi'(z). \tag{1.120}$$

Making use of the conformal mapping function $z = \omega(\eta)$, we can rewrite (1.120) as

$$K = 2\sqrt{2\pi}\lim_{\eta\to\eta_1}(\omega(\eta) - \omega(\eta_1))^{1/2}\frac{\Phi'(\eta)}{\omega'(\eta)}. \tag{1.121}$$

The mapping of a crack of length $2a$ onto a unit circle is given by

$$z = \omega(\eta) = {}^1/_2\, a\,(\eta + \eta^{-1}). \tag{1.122}$$

In this case Eq. (1.121) assumes the simplified form

$$K = 2\,(\pi/a)^{1/2}\,\Phi'(1). \tag{1.123}$$

To illustrate the way in which the results of Muskhelishvili can be used, let us consider two problems solved by the latter.

Let the force F(P, Q) be applied at an arbitrary angle to the side surface of a crack as shown in Fig. 21. The function is given by

$$\Phi(\eta) = \frac{Fa}{4\pi(a^2 - b^2)^{1/2}}\left\{-\frac{1}{\eta} + \left(\frac{\eta_0}{\eta - \eta_0}\right)\left[\left(\eta + \frac{1}{\eta}\right) - \left(\eta_0 + \frac{1}{\eta_0}\right)\right] + \left(\eta_0 - \frac{1}{\eta_0}\right)\times \left[\frac{\varkappa}{1+\varkappa}\log\eta - \log(\eta_0 - \eta)\right]\right\} \tag{1.124}$$

where η_0 corresponds to $z = b$, $F = P - iQ$; $\varkappa = 3 - 4\nu$ in the case of planar straining and $\varkappa = (3 - \nu)/(1 + \nu)$ in the case of a planar stress state.

From (1.123) and (1.124) we obtain

$$K_I = \frac{P}{2(\pi a)^{1/2}}\left(\frac{a+b}{a-b}\right)^{1/2} + \frac{Q}{2(\pi a)^{1/2}}\left(\frac{\varkappa - 1}{\varkappa + 1}\right)$$

$$K_{II} = -\frac{P}{2(\pi a)^{1/2}}\left(\frac{\varkappa - 1}{\varkappa + 1}\right) + \frac{Q}{2(\pi a)^{1/2}}\left(\frac{a+b}{a-b}\right)^{1/2}. \tag{1.125}$$

From (1.125) we find that in the case of a single straight-line crack in an infinite plate with the known stresses $\sigma_y(x, 0)$, $\tau_{xy}(x, 0)$ acting at the side surface of the crack we have

$$K_I = \frac{1}{(\pi a)^{1/2}}\int_{-a}^{a}\sigma_y(x, 0)\left(\frac{a+x}{a-x}\right)^{1/2}dx,$$

$$K_{II} = \frac{1}{(\pi a)^{1/2}}\int_{-a}^{a}\tau_{xy}(x, 0)\left(\frac{a+x}{a-x}\right)^{1/2}dx. \tag{1.126}$$

Let us consider the problem of a crack in the shape of a circular arc of radius R subtending the angle 2α. The crack is symmetric with respect to the x-axis in an infinite plane under uniform tension by the stresses σ at infintiy (Fig. 22).

In this case

$$\Phi'(z) = \frac{\sigma\sqrt{R}}{2(1 + \sin^2 0{,}5\,\alpha)}\times \left\{\frac{z/R - \cos\alpha}{(1 - 2zR^{-1}\cos\alpha + z^2/R^2)^{1/2}} + \sin^2\frac{\alpha}{2}\right\}. \tag{1.127}$$

The crack tip can be shifted to the origin by setting

$$z/R = ie^{i\alpha}(z/R - i - \sin\alpha\cos\alpha) \tag{1.128}$$

whereupon (1.120), (1.127), and (1.128) yield

$$K_I = \frac{\sigma\sqrt{\pi R}}{(1 + \sin^2 0{,}5\,\alpha)}\left(\frac{\sin\alpha\,(1 + \cos\alpha)}{2}\right)^{1/2}$$

$$K_{II} = \frac{\sigma\sqrt{\pi R}}{(1 + \sin^2 0{,}5\,\alpha)}\left(\frac{\sin\alpha\,(1 - \cos\alpha)}{2}\right)^{1/2}. \tag{1.129}$$

Further examples of this procedure can be given.

II. OTHER MODELS OF CRACK PROPAGATION

(The present section is selective in its coverage. The numerous papers concerning the various models of fracture theory would require a separate survey.)

Advances in quasi-brittle fracture theory have been paralleled by the elaboration of other aspects of macroscopic crack development and fracture.

Crack propagation with allowance for redistribution of stresses due to plastic straining was investigated by McClintock [1, 2] (1965, 1958), Hult [1-3] [1957, 1958), and by Hult and McClintock together [1] (1958).

Let us review the basic results of these studies as they relate to fatigue fracture (see Hult [3] (1958)). The analysis was carried out for the case of an ideally plastic material in longitudinal shear. As we see from Fig. 23, the elasticoplastic boundary $R = f(\alpha)$ in the polar coordinates R and α for $\gamma_\infty < \gamma_s$ (where γ_∞ is the shear at infinity and γ_s is the yield strength in shear) is defined by the equations

$$R = R_0\cos\frac{\pi\alpha}{\pi - \omega}, \quad R_0 = cg(\omega)\left(\frac{\gamma_\infty}{\gamma}\right)^{\frac{2\pi-\omega}{\pi-\omega}}$$

$$g(\omega) = \frac{2\sqrt{\pi}}{\pi - \omega}\,\frac{\Gamma(^3/_2 - \omega/2\pi)}{\Gamma(2 - \omega/2\pi)} \tag{2.1}$$

where c is the crack length (Fig. 23b). The function $g(\omega)$ is shown in Fig. 23c.

The shear strain is given by $\gamma = (R/r)\gamma_s$. In the loading cycle defined by the relation $\gamma_\infty = \gamma_{\infty m} \pm \gamma_\infty a$ the strain amplitude in the direction of the crack ($\alpha = 0$) is of the form

$$\gamma_a = \gamma_s\frac{c}{r}g(\omega)\left(\frac{\gamma_\infty}{\gamma_s}\right)^{\frac{2\pi-\omega}{\pi-\omega}}, \tag{2.2}$$

i.e., the strain amplitude does not depend on the average load.

The fatigue crack propagation criterion is formulated as follows: the crack moves forward by the distance ρ when the accumulated plastic shear strain reaches the value γ_f at the distance ρ in front of the crack.

The number of cycles required for the onset of crack propagation and the initial velocity of crack propagation are given by the relations

$$n_0 = \frac{1}{4}\,\frac{\rho}{c_0}\,\frac{\gamma_s\gamma_f}{\gamma_{\infty a}^2}, \quad \left(\frac{\Delta c}{\Delta n}\right)_0 = 8c\,\frac{\gamma_{\infty a}}{\gamma_s\gamma_f}. \tag{2.3}$$

The results were generalized for the case of large strains. It was noted that the twisting which accompanies crack propagation must increase constantly. If the fracture were described in terms of the Griffith-Irwin model, then the crack which develops under the given boundary conditions would appear to be unstable. However, consideration of the plastic zone in front of the crack (elasticoplastic analysis) indicates a stable crack propagation process.

Similar conclusions were drawn by McClintock [2] in the case of noncyclical loads. Considering an already loaded specimen, McClintock investigated the stress redistribution due to an incised groove. He noted that the resulting crack is stable, i.e. that after the initial increase in crack length it is necessary to increase the stress in order to make it propagate further. The crack can become unstable with further development. McClintock later summarized the basic facts on fatigue crack propagation in his comprehensive paper [3]. The growth of fatigue cracks is occasioned by two phenomena: a) cyclical straining, and b) noncyclical progressive straining at the crack tip. The latter paper includes a survey of earlier studies, formulations of fracture criteria, and an analysis of experimental data.

Leonov and Panasyuk [1] (1959) proposed a new model of crack development in brittle bodies. The relevant problems were later elaborated by Panasyuk [1], Leonov [1], et al.

The basic notions developed in these studies are as follows.

The simplest model of a brittle body is defined as a medium with the following properties: a) the maximum tensile stresses do not exceed the rupture strength σ_0; b) the relationship between the stresses and strains is described by Hooke's law if the tensile stresses do not exceed σ_0; c) cracks form in the model when the maximum stress as determined by the methods of linear elasticity theory exceeds σ_0; d) the crack surfaces are drawn together by the stress σ_0 if the gap between them does not exceed some value δ; otherwise, the crack banks do not interact.

The quantities δ, σ_0, and $\mathcal{G}$ are related by the expression

$$\delta\sigma_0 = \mathcal{G} \tag{2.4}$$

where $\mathcal{G}$ is either the fracture energy or the work expanded on enlargement of the crack per unit area.

The range in which the gap between the slit surfaces exceeds δ is called the "ruptured-bond zone" or "crack". The remaining portions of the slit are called the "weakened-bond zones". Fracture (i.e. local fracture) is defined as the conversion of points of the weakened-bond zone into points of the ruptured-bond zone.

Figure 24 shows a straight-line crack in a plane stretched by the tensile stresses σ. The total length of the slit is $2L$; that of the crack is $2l$. The weakened-bond zone at each crack edge is of the length $d = L - l$.

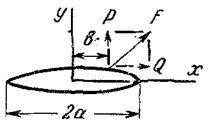

Fig. 21

Problems on the appearance and propagation of cracks can be posed within the framework of the Leonov-Panasyuk model.

Let us consider the planar problem for a straight-line crack in a plane stretched by the stress σ (Fig. 24).

The boundary conditions are of the form

$$\sigma_y(x, \pm 0) = \begin{cases} 0, & |x| \leqslant l \\ \sigma_0, & l \leqslant |x| \leqslant L. \end{cases} \tag{2.5}$$

The limiting tensile stress corresponding to the onset of crack development is given by the formula

$$\sigma = \frac{2}{\pi}\sigma_0 \arccos\exp(-c/l), \qquad c = \frac{\pi E\delta}{8(1-\nu^2)\sigma_0}. \tag{2.6}$$

The length of the weakened-bond zone prior to fracture is

$$d = L - l = l(\exp(c/l) - 1). \tag{2.7}$$

Griffith's formual (1.31) follows from (2.6) as $l \to \infty$.

The Leonov-Panasyuk model is original and is not reducible to the Griffith-Irwin model. The fracture condition is related to the length $d = L - l$, which is generally large. Essentially, those concepts which have been associated with the "Dugdale hypothesis" (see below) in the West were originated by Leonov and Panasyuk.

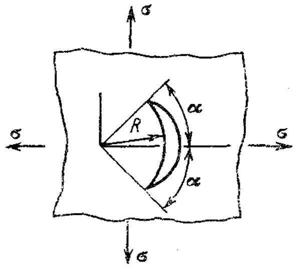

Fig. 22

On the basis of experiments on stretched plates weakened by inner and outer notches (Fig. 25) Dugdale [1] (1960) suggested that the plastic zone is concentrated in a narrow strip over the length of the notch. He obtained the relation

$$\frac{s}{a} = 2\sin^2\left(\frac{\pi T}{4Y}\right), \qquad a = l + s \tag{2.8}$$

where s is the length of the yield zone, l is either the half-length of the inner slit or the length of the outer notch, T is the tensile stress, and Y is the yield strength in tension.

Field [1] (1963) used Dugdale's hypothesis to determine the extent of the plastic zone in a plate with a transverse crack in longitudinal shear.

Goodier and Field [1] (1963) used the same hypothesis to consider work dissipation during the plastic straining which accompanies crack development. They assume that a plane stretched by the forces σ_y is weakened by a crack of length $2l$. The plastic zone near each crack edge is of the length $s = a - l$ (Fig. 25a). Denoting the work of plastic straining by W_p, the authors obtain

$$\frac{1}{4}\frac{dW_p}{dl} = \frac{1}{\pi}(\varkappa + 1)(1 + \nu)E\left(\frac{\sigma_y}{F}\right)^2 lf(t)$$

$$f(t) = \frac{\pi t}{2}\,\mathrm{tg}\,\frac{\pi t}{2} - \log\sec\frac{\pi t}{2}, \qquad t = \sigma_y/\sigma_s \tag{2.9}$$

where $\varkappa$ is a constant in the Muskhelishvili formulas [1] and σ_s is the yield strength in tension.

They then determine the strains at the crack tip, discuss dynamic crack propagation, and compare their findings with experimental data.

Let us also note paper [1] by Lukashev (1963) who developed notions close to those of Leonov-Panasyuk and Dugdale.

Using the Dugdale hypothesis as their starting point, Keer and Mura [1] considered a cracked plane in shear and a circular crack in space with uniform pressure applied to its side surfaces. They also made use of the Tresca plasticity condition.

Problems of plastic zone determination during crack propagation were dealt with on the basis of notions close to those of Dugdale by Rosenfield, Dai, and Hahn [1]. Such problems are also discussed by Rice [2].

Kostrov and Nikitin [1] (1967) recently obtained the solution for a longitudinal shear crack on the basis of the Dugdale hypothesis, requiring fulfillment of the von Mises plasticity condition at the boundary of the plastic zone.

The two-dimensional elasticoplastic stress distribution near the tip of a planar crack is discussed in paper [1] by Swedlow, Williams, and Yang.

In his paper [7] (1962) Williams surveys theoretical and experimental studies on the fracture of elasticoplastic media. He notes that one of the earliest studies on the fracture of elasticoplastic materials was that of Rivlin and Thomas (J. Polymer Sci., Vol. 10, 1952) who used the

Griffith hypotheses to study the rupture of rubber sheets. These authors obtained a rupture criterion similar to that of Griffith. Greensmith and Thomas (J. Polymer Sci., Vol. 18, 1955) later found that the critical rate of elastic energy liberation corresponding to the onset of fracture depends on the velocity of rupture and temperature.

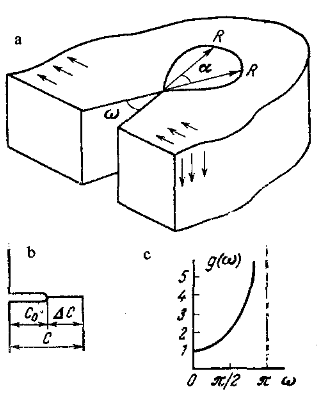

Fig. 23

As in his previous study [6], Williams used the Voigt model of a viscoelastic body as his starting point, formulating the viscoelastic analysis problem as one of determining the time until onset of crack propagation.

Kachanov [1, 2] (1961, 1963) considered crack propagation in Maxwell-type viscoelastic media, in hereditary media, etc., assuming a constant stress intensity coefficient.

Berg [1] (1962) considered the development of planar elliptical cracks in a viscous medium. Using the apparatus developed by Muskhlishvili, he investigated cases of crack development under various boundary conditions and compared his findings with experimental data.

Barenblatt, Entov, and Salganik [1, 2] (1966) considered an elastic medium under the assumption that the stress intensity coefficient depends on the velocity of crack progression.

McClintock and Irwin [1] (1965) discussed the effects of plasticity on fracture and the possible refinements of quasi-brittle fracture theory in connection with allowance for the plastic properties of the material. Their study primarily concerns pure shear. They show that energy dissipation rate during plastic straining is equal to double the value of the $\mathscr{G}$ obtained from linear elasticity analysis. In the case of plasticity none of the criteria of the $\mathscr{G}$-const type is adequate for describing fracture. The fracture criterion can be based, however, on the local stresses and strains in some domain in front of the crack. A fracture criterion based on the displacements near the crack tip during its opening is generally inconsistent with one based on the local stress and strain characteristics in front of the crack. The authors then proceed to a detailed discussion of stable crack growth, devoting much attention to the analysis of experimental data.

We must also note the cycle of original studies begun by Leonov and coworkers in 1961. These studies concern the strain state of elastic materials weakened by dislocations and plastic slip bands.

Recent contributions include a study by Erdogan [5] (1966) on the determination of the plastic zone in a crack-weakened semi-infinite medium consisting of two heterogeneous parts in shear, as well as papers by Yokobori and Ichikawa [1] (1965) and Yakobori and Tnaka [1] (1966). In his paper [8] (1967) Cherepanov used general energy considerations to discuss problems of crack propagation in various media and to investigate crack development criteria in plastic and viscoelastic bodies.

III. SOLUTIONS OF PROBLEMS OF THE MATHMATICAL THEORY OF CRACKS

The present section is a brief survey of solutions of the mathematical theory of cracks based on the linear elastic body model. (The reader will find additional information on the papers cited in RZh Mekhanika and also in Donaldson and Anderson [1], Paris and Sih [1], Barenblatt [9, 11], and the recent survey on stress concentration theory by Neuber and Hahn [1].)

Soviet researchers have made some outstanding contributions to the theory of elasticity. There is no need here to give a detailed account of studies on various methods of solving boundary value problems and stress concentration theory which form the basis of many studies on quasi-brittle fracture theory. Suffice it to say that the contributions of Kolosov, Muskhelishvili, and several other investigators have made possible the solution of basic problems of the mathematical theory of cracks.

In addition to his general methods, which have played a major role in the development of the planar problem of elasticity theory, Muskhelishvili also solved several problems on cracked elastic bodies.

The well-known studies of L. A. Galin, S. G. Lekhnitskii, A. I. Lur'e, G. N. Savin, Ya. S. Uflyand, D. I. Sherman, I. Ya. Shtaerman et al. have a direct bearing on the mathematical theory of quasi-brittle fracture. The results of Vorovich and his pupils on the contct problem in bodies of finite size have been applied to crack theory.

Results obtained in related fields of mechanics by M. V. Keldysh, N. E. Kochin, M. A. Lavrent'ev, L. I. Sedov et al. have been used and will be used in the future to solve problems of quasi-brittle fracture theory.

§1. An isotropic elastic body. The planar problem. We have already mentioned the studies of Kolosov, Inglis, Muskhelishvili, Wolf, Neuber [1], Westergaard [1, 2], Sneddon [1–3], Irwin [3–5, 7, 9], et al. These authors investigated a broad range of problems relating to an infinite domain weakened by one or more cracks.

Willmore [1] investigated a plane weakened by a straight-line crack with arbitrary loads applied to its banks. He also considered the case of two collinear cracks of the same length in an infinite plane under homogeneous tension. This case was later considered by Winne and Wundt [1].

Bowie [1] investigated the stress state of a plane weakened by a circular hole with radial cracks extending from the boundary of the latter. He considered two types of loads, namely omnilateral and uniaxial tension. He also studied the effect of a hole on the stress state near a crack.

Sadowsky [1] discussed once again the analogy between die impression and crack problems and solved the problem of two collinear cracks in a plane under omnilateral tension at infinity.

The mixed problem of a plane with a slit was considered by Mossakovskii and Zagubizhenko [1, 2]. The opening of a crack by a wedge was investigated by Barenblatt [5], Barenblatt and Cherepanov [2], Markuzon [1], and Cherepanov [3].

Koiter [1] studied the problem of a plane weakened by a row of collinear cracks in shear (which cannot be solved by the method of Westergaard, since $\tau_{xy} \neq 0$ for y = 0). Koiter [2] likewise obtained an approximate solution of the problem of a plane weakened by an infinite row of parallel equidistant cracks of equal length, assuming the presence of shearing forces at infinity. A plane weakened by an infinite row of cracks was aslo studied by Lowengrub [1].

Bueckner [2] solved several crack problems, including that of a single crack emerging onto the boundary of a circular hole with arbitrarily varying normal forces applied to the side surface of the crack.

A major cycle of studies on the planar problem with cracks was carried out by the Ukrainian researchers Panasyuk, Babich, Berezhnitskii, Buina, Kaminskii, Kovchik, Libatskii, Lozov, et al., who considered the solutions of various problems for a plane weakened by several cracks and hole-crack combinations, as well as the bending of cracked strips, etc.

The studies of Barenblatt and Cherepanov [1], Sorokin [1], Ustinov [1] et al. must also be mentioned.

The cases of one and two collinear cracks in a plane with various combinations of boundary conditions were studied by Erdogan [1]. Noteworthy too is the paper by Sih, Paris, and Erdogan [1].

Lowengrub [1] investigated the stress state of a plane with two outer cracks for various combinations of boundary conditions.

Single cracks emerging onto the free boundary of a semi-infinite body were studied by Wigglesworth [3] and Irwin [6]; Wigglesworth [4] also studied the problem of a crack emerging onto the boundary of a circular cavity. Bueckner [2] considered the bending of an elastic strip with a crack emerging onto the strip boundary (Bueckner's study is related to that of Winne and Wundt [1]).

Sneddon's monograph [4] contains a discussion of the two solutions obtained by Tait for strips of finite width weakened by inner and outer symmetric cracks. Symmetric forces were assumed to act at the side surfaces of the cracks.

Determinations of the stress states of strips of finite width was considered by Aleksandrov [1] and by Aleksandrov and Smetanin [1].

Practically important solutions for stretched strips of finite width containing central cracks were obtained by Isida [1], who investigated stress concentrations at the vertices of cracks with rounded tips. The results for a crack are obtained by the limiting process (1.114). An approximate solution (1.74) was likewise obtained by Irwin [3]. The Isida solution is suitable for the treatment of experimental data on the constants $\mathcal{G}_c$.

Isida [3] also considered the problem of a finite strip with an eccentric crack.

Bowie studied the stretching of a strip with two symmetric outer cracks. He also computed stress concentration coefficients and made comparisons with the results of Irwin [3] and Bueckner [2] in several different cases.

Bowie made extensive use of the results of Kartsivadze [1].

Bowie and Neal [1] considered the problem of stretching of a rectangular plate with a single outer crack.

Bloom [1] used the methods developed by Bowie to solve the problem of bending of a rectangular strip weakened by a single crack. Knauss [1] considered the determination of the stress state of a strip containing an infinite crack along its center line. He assumed that the side edges of the strip experienced a rigid displacement normal to the crack.

Lowengrub [3] considered the problem of a strip of finite width weakened by a center-line crack parallel to the sides of the strip under various boundary conditions imposed on the stresses and displacements.

Solutions of problems concerning cracked structures strengthened by reinforcing ribs are of practical interest. One of the first studies in this area was that of Romualdi and Sanders [1]. Isida [2] generalized the results of the latter authors to consider problems of centrally cracked strips with reinforced edges, as well as of infinite sheets with a periodic row of cracks reinforced by longitudinal stiffening elements.

Greif and Sanders [1] investigated the problem of a plane weakened by an asymmetric crack and reinforced by an infinite stringer. Bloom and Sanders [1] considered a similar problem for a stringer fastened to an infinite sheet by means of rigid equidistant rivets of equal diameter.

There have been several other foreign studies in the field. (I was unable to gain access to the papers of Sanders (Report NACA, 1959); Romualdi, Fraser, Irwin (Report NACA, 1957); Leybold (Technical Note NACA, 1963) et al.)

§2. Axisymmetric and three-dimensional problems. The well-known solutions of Neuber [1] have already been mentioned. Sack [1] used Neuber's results to compute the Griffith critical stress for a space weakened by a penny-shaped crack under uniform tensile forces at infinity.

Sneddon [1, 2] obtained the general solution of the problem of a circular crack with normal forces acting at its side surfaces. Payne [1] and Green and Zerna [1] also investigated axisymmetric problems for a space with a circular crack.

Axisymmetric problems for a space weakened by circular slits were considered by Uflyand [1, 2] and by Lebedev and Uflyand [1].

The axisymmetric problem for a half-space with a circular slit was studied by Kuz'min and Uflyand [1].

A circular crack in an unbounded medium in a homogeneous shear field was investigated by Westmann [1].

Problems of determining the stress state of a long circular cylinder containing a symmetrically situated circular crack perendicular to its axis were dealt with by Collins [1], Sneddon and Tait [1], and Sneddon and Wells [1] under various combinations of boundary conditions.

Erdogan [4] considered the case of a plane weakened by circular annular cracks. He obtained his results for two interconnected elastic half-spaces made of materials with isotropic but differing properties. He assumed that given forces acted at the side surfaces of the cracks and considered inner and outer axisymmetric slits.

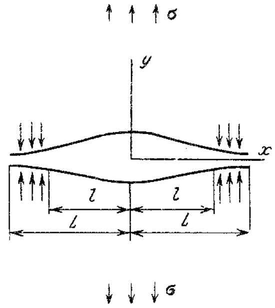

Fig. 24

Lowengrub and Sneddon [1] considered the case of an outer axisymmetric slit under various axisymmetric and nonaxisymmetric boundary conditions.

The three-dimensional problem of stress concentration in an infinite body weakened by a cavity in the shape of a triaxial ellipsoid was investigated by Sadowsky and Sternberg [1] and Green and Sneddon [1]. These authors considered loads applied symmetrically with respect to the principal planes of the ellipsoidal plane or with respect to the plane of a two-dimensional crack.

The three-dimensional problem with arbitrary loading of a circular slit was investigated by Mossakovskii [1]. Several three-dimensional problems were solved by Panasyuk [4–6, 9, 12], who considered the cases of a space with an elliptical crack, and of a nearly circular crack acted on by a load normal to the crack banks.

Irwin's study [9] has been mentioned. Kassir and Sih [1] considered the solution for an unbounded space with a planar elliptical crack acted on by shearing loads. By superposition of the solution for the case of tension perpendicular to the crack plane they solved the problem of an unbounded elastic body with a planar elliptical crack under a load of general form.

§3. Torsion, longitudinal shear, bending. One of the first authors to solve a problem on the torsion of a rod with limiting slits was Filon [1] (1900).

Problems concerning the torsion of cracked bodies were studied by Dinnik [1], who in 1913 solved a problem for a circular rod with a radial crack. Other papers on the subject include those of Arutyunyan and Abramyan [1], Balobyan [1], Shiryaev [1, 2], and Wigglesworth [1, 2].

Irwin's results [5, 7] for the case of longitudinal shear have already been mentioned.

The case of longitudinal shear was also investigated by Barenblatt and Cherepanov [3] and by Salganik [1].

Sih [4] studied problems concerning the determination of stress concentration coefficients in the bending and torsion of cracked beams. He investigated the same problems in the case of longitudinal shear in his papers [6, 7].

Anisotropic materials. A crack in an orthotropic material was considered by Willmore [1]. Chapkis and Williams [1] considered the behavior of the solution near a crack tip in an orthotropic plate. Stroh [1] considered a straight-line crack in an anisotropic plane. He obtained the basic solution for the case of forces distributed along the side surfaces of the crack, investigated the properties of the solution at the crack tip, and computed the change in elastic energy associated with a change in crack length. Anisotropic materials containing cracks were considered by Barenblatt and Cherepanov [4]. The study by Ang and Williams [1] is also worthy of mention. Problems involved in determining stress concentration coefficients for anisotropic media were discussed by Paris and Sih [1]. The same problems are considered in more detail in paper [1] by Sih, Paris, and Irwin.

Nonhomogeneous materials. The problem of two half-planes or half-spaces of differing materials with cracks at the contact surface has been investigated by several authors. Nonhomogeneous cracked materials were studied by Cherepanov [1], Mossakovskii and Rybka [1], Salganik [2], and Gol'dshtein and Salganik [1]. The most general results were obtained by Cherepanov [1, 3]. The behavior of the solution at the crack vertex in this case was investigated by Williams [4].

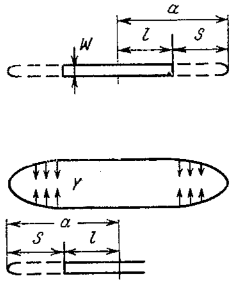

Fig. 25

Several boundary value problems in the planar case were investigated by Erdogan [2, 3]. The studies of Williams and Erdogan showed that the singularity of the solution near the crack tip is of the form

$$\sigma_{ij} = \frac{k}{(2\pi r)^{1/2}} f_{ij}((\lambda_i), \theta, \log r) \qquad (3.1)$$

where λ_i is a function of the material properties.

Erdogan's study [4] for the axisymmetric case has already been mentioned.

Sack and Williams [1] considered the properties of the solution near a crack perpendicular to the boundary between two media. It turns out that in this case $\sigma_{ij} \sim r^{-n}$ (the singularity exponent $n > 1/2$) near the crack if the crack arose in the material with the lower elastic modulus, and vice versa.

England [1] studied the displacement field in the problem of a single straight-line crack situated along the line of contact between two half-planes of different materials. Assuming that normal forces act at the side surfaces of the crack, he showed that the lower and upper edges of the crack bend and overlap near its tip, which is physically impossible.

Several solutions of the planar problem of cracks at the boundary between two heterogeneous media were obtained by Rice and Sih [1]. The studies of England [3] and Sih [9] are also noteworthy.

Bending of plates and shells. Williams [5] investigated the properties of solutions near crack tips in bent plates. Knowles and Wang [1] used Reisner's theory to determine the stress state of a thick crack-weakened plate in bending. The singularity of the stresses near the crack tip is of the same order as in the classical theory of plates.

The aforementioned study by Sih, Paris, and Erdogan [1] contains a discussion of the determination of stress intensity coefficients in bending and several examples. The bending stresses in a cracked plate resting on an elastic base were considered by Ang, Folias, and Williams [1], who pointed out the analogy between their problem and that of the straining of a spherical shell with a small initial curvature.

Sih and Rice [1] considered the bending of a plate consisting of two heterogeneous parts joined along a straight line with cracks along it.

The paper by Redwood and Shepherd [1] has already been mentioned.

Survey [1] by Paris and Sih contains several additional references to studies by foreign authors which were not available to me.

Thermal stresses. Sih's paper [2] was mentioned above. The thermal problem for a space with a circular slit was considered by Sneddon [4]. Florence and Godie [1] obtained the solution for a plane weakened by an oval hole in a steady temperature field. Other examples are mentioned by Paris and Sih [1]. Thermal problems were also studied by Borodachev [1], Kit and Podstrigach [1], et al.

Moment stresses. Paris and Sih [1] mention a doctoral thesis by Setzer (1963) in which crack problems are investigated by means of the moment theory of elasticity.

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